Boussinesq Hypothesis

- Model the Reynolds stress like the viscous stress, but with a turbulent viscosity:

$$-\rho\overline{u_i'u_j'} = 2\mu_t S_{i,j} - \frac{2}{3}\rho k \delta_{i,j}$$

$$k = \frac{1}{2}\overline{u_i'u_i'}$$

$$S_{i,j} = \frac{1}{2}\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right] - \frac{1}{3}\frac{\partial U_k}{\partial x_k}\delta_{i,j}$$

- Then model the turbulent viscosity $\Gamma = \mu_t/\rho$.
- Prandtl Mixing Length (0 equation model)

$$\Gamma (=) \frac{m^2}{s} \sim \frac{l^2}{\tau} \rightarrow C|u'|l_m$$

  - l_m is a mixing length (specified), |u'| is a turbulence intensity (or some other characteristic velocity scale).
  - Prandtl used (Turns): $\Gamma = 0.1365\, l_m(\bar{v}_{x,\,max} - \bar{v}_{x,\,min})$

$$l_m = 0.075\delta_{99\%}$$ In jets, other relations for other flows

One Equation Model

- Prandtl's one equation model:
  - http://www.cfd-online.com/Wiki/Prandtl%27s_one-equation_model

$$\Gamma (=) \frac{m^2}{s} \sim |u'|l \sim k^{1/2} l = C\frac{k^2}{\epsilon}$$

  - Solve k from the kinetic energy transport equation:

$$\underbrace{\frac{\partial k}{\partial t}}_{\text{transient}} + \underbrace{\frac{\partial (k u_i)}{\partial x_i}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\Gamma}{\sigma_k}\right)\frac{\partial k}{\partial x_i}\right]}_{\text{diffusion}} \underbrace{- \overline{u_i'u_j'}\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)}_{\text{production (P)}} \underbrace{- \epsilon}_{\text{dissipation}}$$

  - Model ε as

$$\epsilon = C\frac{k^{3/2}}{l_m}$$

  - $\sigma_k = 1$, l_m is the specified (by user) turbulence length scale.

Two-Equation Model: k-ε

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- The standard turbulence model for RANS
- As before:

$$\Gamma (=) \frac{m^2}{s} \sim |u'| l \sim k^{1/2} l = C \frac{k^2}{\epsilon}$$

- Transport equations for k, and ε:

$$\frac{\partial k}{\partial t} + \frac{\partial (k u_i)}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\Gamma}{\sigma_k}\right)\frac{\partial k}{\partial x_i}\right] \underbrace{- \overline{u_i' u_j'}\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)}_{P_r} - \epsilon$$

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial (\epsilon u_i)}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\Gamma}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_i}\right] + C_{1\epsilon}\frac{\epsilon}{k}P_r - C_{2\epsilon}\frac{\epsilon^2}{k}$$

$$C = 0.09,\ \sigma_k = 1,\ \sigma_\epsilon = 1.3,\ C_{1\epsilon} = 1.44,\ C_{2\epsilon} = 1.92$$

Use of k-ε Model

- Fairly easy to use (2 transport equations)
- Does as well as most other models over a broad range
- Can be adjusted for swirl
- 5 constants required (adjusted)
- Non-linear k-ε version (add extra terms) used at BYU
- RNG k-ε used in Fluent (add extra terms)
- Need boundary conditions
  - Flows become laminar near walls
  - Most use universal law of the wall parameters
    - y^+ and u^+ related by logarithmic correlation

k-ε properties, Other versions

- Standard k-ε
  - Poor at high Δp
  - Poor for swirling flows
  - ε equation is semi-empirical
  - Fully developed flows
- RNG k-ε
  - Extra terms in ε equation
  - Better for swirling flows
  - Better at lower Re
  - Swirl modifications
- Realizable k-ε
  - Newer (1995)
  - Exact ε equation
  - Better jet spreading rates
- k-ω model
  - ω is the specific dissipation rate, with a transport equation (replaces ε)
  - Several versions

2-Equation Variants

We return to the length scale equation:

$$l_\varepsilon = \frac{k^{\frac{3}{2}}}{\varepsilon}. \tag{7.56}$$

If we wish to derive a 'transport' equation for this length scale itself, we can do so as follows:

$$\frac{Dl_\varepsilon}{Dt} = 1.5\frac{k^{\frac{1}{2}}}{\varepsilon}\frac{Dk}{Dt} - \frac{k^{\frac{3}{2}}}{\varepsilon^2}\frac{D\varepsilon}{Dt}. \tag{7.57}$$

This is so stated, but not actually executed, merely to highlight the fact that the length scale may, indeed, be thought of as being transported, generated and destroyed by various interactions that can be described exactly, via the terms in the turbulence-energy and dissipation-rate equations. Our preference for the dissipation rate is due to the fact that it is this quantity that appears as the unknown length-scale surrogate in the turbulence-energy equation.

Equation (7.57) serves to introduce the general fact that any quantity of the form:

$$\phi = k^m \varepsilon^n, \tag{7.58}$$

can serve as a length-scale surrogate, and can be derived from:

$$\frac{D\phi}{Dt} = \varepsilon^n m k^{(m-1)}\frac{Dk}{Dt} + k^m n \varepsilon^{(n-1)}\frac{D\varepsilon}{Dt}. \tag{7.59}$$

The most popular alternative to ε itself is $\phi = \varepsilon/k$, the 'specific dissipation', usually denoted by ω.[10] The reason this alternative should

Table 9.1: Alternative length-scale-surrogate variables and their near-wall behaviour.

ϕ	Dimensions	$\nu_t \propto \ldots$	Wall-asymptotic behaviour	Wall value
ε	$[\mathrm{L}^2/\mathrm{T}^3]$	$\frac{k^2}{\varepsilon}$	$O(1)$	$2\nu\frac{k}{y^2}$
$\omega \equiv \frac{\varepsilon}{k}$	$[\mathrm{T}^{-1}]$	$\frac{k}{\omega}$	$O(y^{-2})$	∞
$\omega^2 \equiv \left(\frac{\varepsilon}{k}\right)^2$	$[\mathrm{T}^{-2}]$	$\frac{k}{\sqrt{\omega^2}}$	$O(y^{-4})$	∞
$\tau \equiv \frac{k}{\varepsilon}$	$[\mathrm{T}]$	$k\tau$	$O(y^2)$	0
$kL \equiv \frac{k^{5/2}}{\varepsilon}$	$[\mathrm{L}^3/\mathrm{T}^2]$	$(kL)/k^{1/2}$	$O(y^5)$	0
$L \equiv \frac{k^{3/2}}{\varepsilon}$	$[\mathrm{L}]$	$k^{1/2}L$	$O(y^3)$	0
$\nu_t \equiv C_\mu\frac{k^2}{\varepsilon}$ or $R_t \equiv \frac{\nu_t}{\nu}$	$[\mathrm{L}^2/\mathrm{T}]$	ν_t	$O(y^4)$	0
$\varsigma \equiv \frac{\varepsilon}{k^{1/2}}$	$[\mathrm{L}/\mathrm{T}^2]$	$\frac{k^{3/2}}{\varsigma}$	$O(y^{-1})$	∞

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Reynolds Stress Model

$$\underbrace{\frac{\partial}{\partial t}(\rho\,\overline{u_i'u_j'})}_{\text{Local Time Derivative}} + \underbrace{\frac{\partial}{\partial x_k}(\rho u_k \overline{u_i'u_j'})}_{C_{ij}\equiv\text{Convection}} = \underbrace{-\frac{\partial}{\partial x_k}\left[\rho\,\overline{u_i'u_j'u_k'} + \overline{p\left(\delta_{kj}u_i' + \delta_{ik}\right.}\right.}_{D_{T,ij}\equiv\text{Turbulent Diffusion}}$$

$$+ \boxed{\underbrace{\frac{\partial}{\partial x_k}\left[\mu\frac{\partial}{\partial x_k}(\overline{u_i'u_j'})\right]}_{D_{L,ij}\equiv\text{Molecular Diffusion}}} \underbrace{-\rho\left(\overline{u_i'u_k'}\frac{\partial u_j}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial u_i}{\partial x_k}\right)}_{P_{ij}\equiv\text{Stress Production}} \underbrace{-\rho\beta(g_i\overline{u_j'\theta}}_{G_{ij}\equiv\text{Buoyancy}}$$

$$+ \boxed{\underbrace{\overline{p\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}}_{\phi_{ij}\equiv\text{Pressure Strain}}} - \boxed{\underbrace{2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}_{\epsilon_{ij}\equiv\text{Dissipation}}}$$

$$\underbrace{-2\rho\Omega_k\left(\overline{u_j'u_m'}\epsilon_{ikm} + \overline{u_i'u_m'}\epsilon_{jkm}\right)}_{F_{ij}\equiv\text{Production by System Rotation}} + \underbrace{S_{\text{user}}}_{\text{User-Defined Source Term}} \quad (11.6\text{-}1)$$

Idea:
Write transport equations for each of the 6 Reynolds stress terms

Boxes represent terms that must be modeled

$$\boxed{\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{(u_i'u_j'u_k')} + \frac{\overline{p}}{\rho}(\delta_{kj}u_i' + \delta_{ik}u_j') - \nu\frac{\partial}{\partial x_k}(\overline{u_i'u_j'})\right]}_{\text{Diffusive Transport}} = \frac{\partial}{\partial x_k}\left(\frac{\nu_t}{\sigma_k}\frac{\partial(\overline{u_i'u_j'})}{\partial x_k}\right) \quad (6.2\text{-}15)}$$

Secondly, the pressure-strain term is approximated as[25]:

$$\boxed{\frac{\overline{p}}{\rho}\left[\overline{\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}}\right] = -C_3\frac{\epsilon}{k}\left[\overline{u_i'u_j'} - \frac{2}{3}\delta_{ij}k\right] - C_4\left[P_{ij} - \frac{2}{3}\delta_{ij}P\right] \quad (6.2\text{-}16)}$$

where C_3 and C_4 are empirical constants whose values are $C_3 = 1.8$ and $C_4 = 0.60$, $P = \frac{1}{2}P_{ii}$, and

$$P_{ij} = -\overline{u_i'u_k'}\frac{\partial u_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial u_i}{\partial x_k} \quad (6.2\text{-}17)$$

Finally, the dissipation term in Equation 6.2-14 is assumed to be isotropic and is approximated via the scalar dissipation rate[46]:

$$\boxed{2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} = \frac{2}{3}\delta_{ij}\epsilon} \quad (6.2\text{-}18)$$

Algebraic Stress Model

Idea:
Write algebraic equations for each of the Reynolds stress terms,
In terms of k and ε

Note:
No terms with $\frac{\partial}{\partial x}\left(\overline{u'u'}\right)$

TABLE 7. Algebraic stress model in cylindrical coordinates

$\overline{v'v'}$ Equation

$$\left[1+\frac{4}{3}\lambda\left(\frac{k}{\varepsilon}\right)\frac{\partial\bar{v}}{\partial x}\right]\overline{v'v'}=\frac{2}{3}k+\frac{2}{3}\lambda\frac{k}{\varepsilon}\left\{\left(\frac{\partial\bar{u}}{\partial x}\right)\overline{u'u'}+\frac{\partial\bar{u}}{\partial r}-2\frac{\partial\bar{v}}{\partial x}\right\}\overline{v'u'}$$

$$+\left(\frac{\bar{v}}{r}\right)\overline{w'w'}+\left[\frac{\partial\bar{w}}{\partial r}+\frac{\bar{w}}{r}(2+3\beta)\right]\overline{v'w'}+\left(\frac{\partial\bar{w}}{\partial x}\right)\overline{w'u'}$$

$\overline{u'u'}$ Equation:

$$\left[1+\frac{4}{3}\lambda\left(\frac{k}{\varepsilon}\right)\frac{\partial\bar{u}}{\partial x}\right]\overline{u'u'}=\frac{2}{3}k+\frac{2}{3}\lambda\frac{k}{\varepsilon}\left\{\left(\frac{\partial\bar{v}}{\partial r}\right)\overline{v'v'}+\left(\frac{\partial\bar{v}}{\partial x}-2\frac{\partial\bar{u}}{\partial r}\right)\overline{v'u'}\right.$$

$$\left.+\left(\frac{\bar{v}}{r}\right)\overline{w'w'}+\left(\frac{\partial\bar{w}}{\partial r}-\frac{\bar{w}}{r}\right)\overline{v'w'}+\left(\frac{\partial\bar{w}}{\partial x}\right)\overline{w'u'}\right\}$$

$\overline{v'u'}$ Equation:

$$\left[1-\lambda\left(\frac{k}{\varepsilon}\right)\frac{\bar{v}}{r}\right]\overline{v'u'}=\lambda\left(\frac{k}{\varepsilon}\right)\left[-\left(\frac{\partial\bar{u}}{\partial r}\right)\overline{v'v'}-\left(\frac{\partial\bar{v}}{\partial x}\right)\overline{u'u'}+\left(\frac{\bar{w}}{r}\right)(1+\beta)\overline{w'u'}\right]$$

$\overline{w'w'}$ Equation:

$$\left[1+\frac{4}{3}\lambda\left(\frac{k}{\varepsilon}\right)\frac{\bar{v}}{r}\right]\overline{w'w'}=\frac{2}{3}k+\frac{2}{3}\lambda\left(\frac{k}{\varepsilon}\right)\left\{\left(\frac{\partial\bar{v}}{\partial r}\right)\overline{v'v'}+\left(\frac{\partial\bar{u}}{\partial x}\right)\overline{u'u'}+\left(\frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{u}}{\partial r}\right)\overline{v'u'}\right.$$

$$\left.-\left[2\frac{\partial\bar{w}}{\partial r}+(1+3\beta)\frac{\bar{w}}{r}\right]\overline{v'w'}-2\left(\frac{\partial\bar{w}}{\partial x}\right)\overline{w'u'}\right\}$$

$\overline{v'w'}$ Equation:

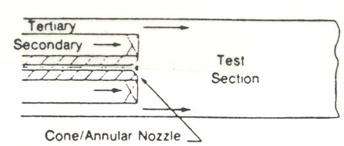
$$\left[1-\lambda\left(\frac{k}{\varepsilon}\right)\frac{\partial\bar{u}}{\partial x}\right]\overline{v'w'}=\lambda\left(\frac{k}{\varepsilon}\right)\left\{-\left(\frac{\partial\bar{w}}{\partial r}+\beta\frac{\bar{w}}{r}\right)\overline{v'v'}-\left(\frac{\partial\bar{w}}{\partial x}\right)\overline{v'u'}+\left[(1+\beta)\frac{\bar{w}}{r}\right]\overline{w'w'}-\left(\frac{\partial\bar{v}}{\partial x}\right)\overline{w'u'}\right\}$$

$\overline{w'u'}$ Equation:

$$\left[1-\lambda\left(\frac{k}{\varepsilon}\right)\frac{\partial\bar{v}}{\partial r}\right]\overline{w'u'}=\lambda\left(\frac{k}{\varepsilon}\right)\left\{-\left(\frac{\partial\bar{w}}{\partial x}\right)\overline{u'u'}-\left(\frac{\partial\bar{w}}{\partial r}+\beta\frac{\bar{w}}{r}\right)\overline{v'u'}-\left(\frac{\partial\bar{u}}{\partial r}\right)\overline{v'w'}\right]$$

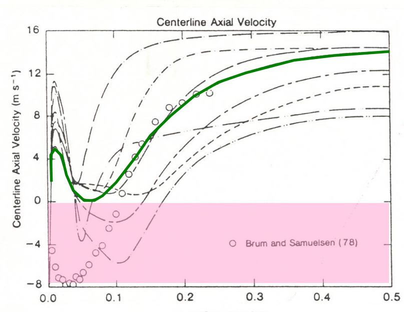
where $\beta=\psi/(1-C_2)$

Example: Non-reacting swirling flow



Cases 1 and 2—Brum and Samuelson[78]

Standard
k-ε



Negative centerline
velocity region

FIG. 4. Comparison of predicted and measured centerline axial and velocity profiles for Case 1 (data from Brum and Samuelsen[78]; legend supplied by Table 18).

TABLE 18. Legend description for case studies

Turbulence model description/legend	Equations of reference	Case reference: Brum and Samuelsen[78] 1	Case reference: Yoon[71] 4	Case reference: Yoon[71] 5	Case reference: Roback and Johnson[164] 6
Standard k–ε model	34–42				
LPS gradient Richardson no.*	108–110	$C_{gs}=0.10$ MTS	$C_{gs}=0.005$ TTS	$C_{gs}=0.03$ MTS	$C_{gs}=0.005$ TTS
Rodi flux Richardson no.*	114, 115	$C_{fs}=0.90$ MTS			$C_{fs}=0.90$ MTS
"Boysan" Richardson no.	109, 112	$C_{gs}=0.20$			
Modified C_μ coefficient	134	$C_a=0.03$ $C_b=2.63$		$C_a=0.03$ $C_b=2.63$	
Gibson–Launder ASM* (I)	76–99	$C_1=2.5$ $C_2=0.55$	$C_1=2.5$ $C_2=0.55$	$C_1=2.5$ $C_2=0.55$	$C_1=2.5$ $C_2=0.55$
Gibson–Launder ASM (II)	76–99	$C_1=2.0$ $C_2=0.40$			
Gibson–Launder ASM (III) added convection	76–99	$\psi=0.40$ $C_1=2.5$ $C_2=0.55$	$\psi=0.40$ $C_1=2.5$ $C_2=0.55$		

*MTS = Mean flow time scale in the denominator of the Richardson number.
TTS = Turbulence time scale in the denominator of the Richardson number.
ASM = Algebraic stress model.

from Sloan, et al.,