Nonlinear Equation 2

Outline

- Fixed point iteration
- Secant method
- Newton's method

Comments

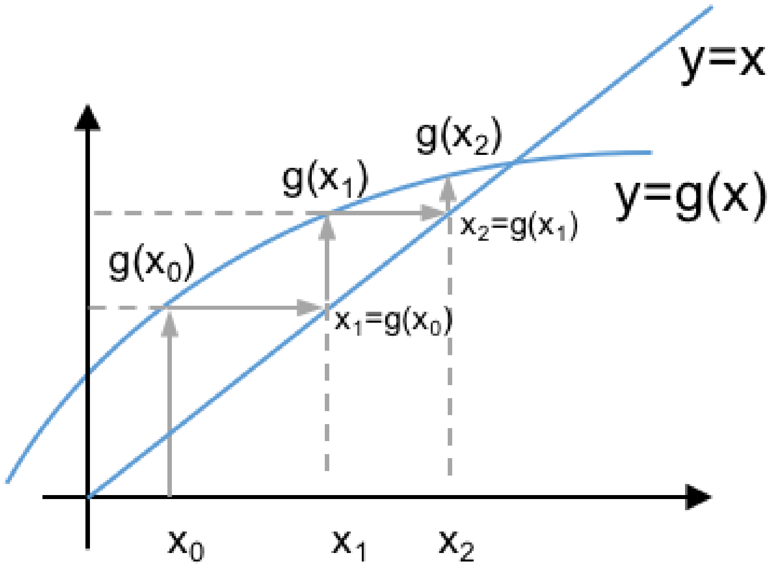
- Focusing on 1 equation and 1 unknown here.
- Methods only require information at one point.
    - Termed "open" methods
    - Unlike the closed (bracketing) methods that required two initial points.
- Methods may converge faster.
- Methods may diverge (the tradeoff).
- Often used to refine a root from a slower method like bisection.

Fixed Point Method

- Very common
- Very simple
- Rewrite $f(x)=0$ as $x=g(x)$.
    - Can always do this: just add x to both sides.
        - But there is often more than one way to do this, and the approch used may affect stability, as shown below.
- Iteration

$$\mathbf{x_{k+1}=g(x_k)}$$

- That is: guess x_0.
- Evaluate $g(x_0)$.
- The result is x_1.
- Repeat.



Convergence depends on

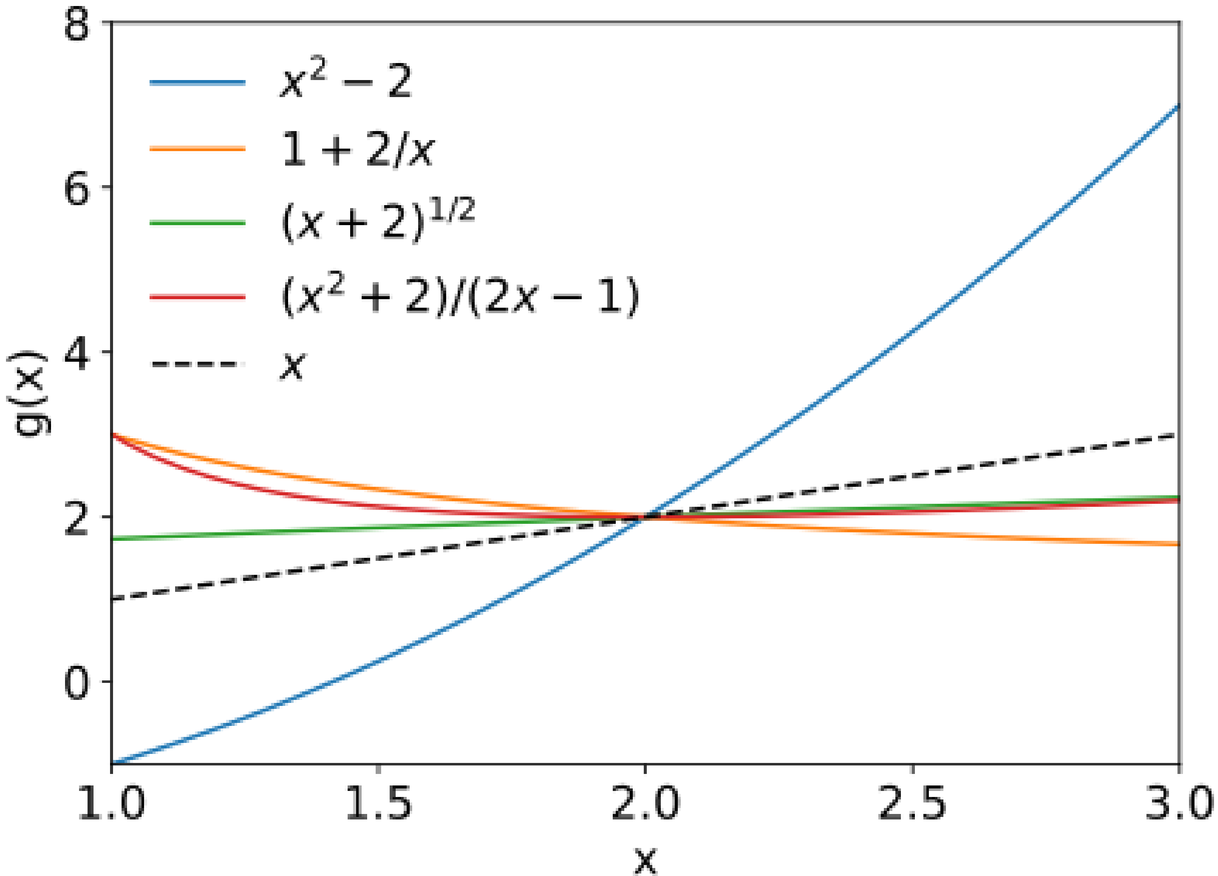
1. *Initial guess*
2. *Form chosen for g(x)*

Example

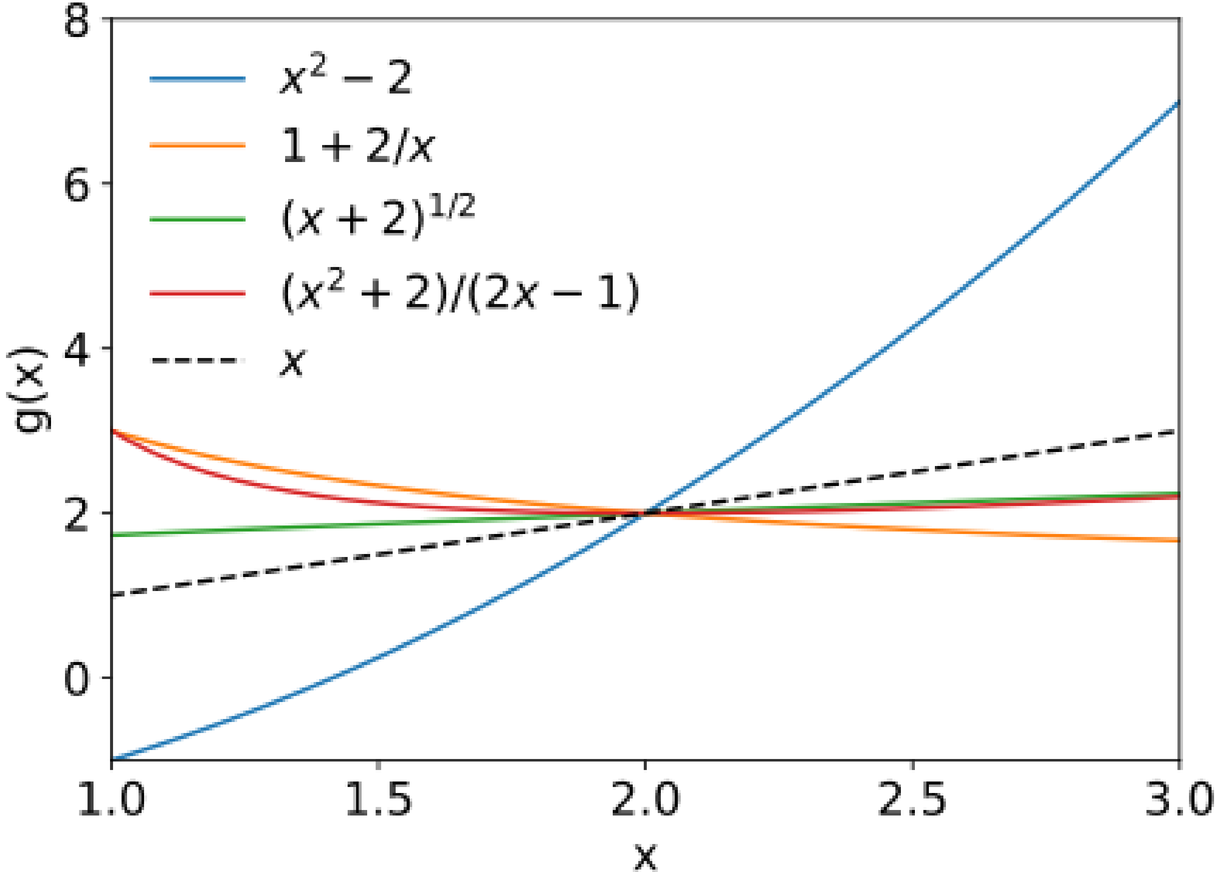
$f(x) = x^2 - x - 2$

- Add x to both sides of f(x)=0: $x = x^2 - 2 = g(x)$
- Add x+2 to both sides of f(x)=0, and divide the result by x: $x = 1 + \frac{2}{x} = g(x)$
- Add x+2 to both sides and then take the square root: $x = \sqrt{x+2} = g(x)$
- Add x^2+2 then divide by (2x−1): $x = \frac{x^2+2}{2x-1} = g(x)$

Example



Example



Example 1

Solve $x = g(x)$ for $g(x) = (x+2)^{1/2}$

Example 2

- Fluid mechanics, turbulent pipe flow
- Given ΔP, D, L, ϵ/D, ρ, μ.
- Find the velocity in the pipe.

$$Re = \frac{\rho D v}{\mu}.$$

$$\frac{1}{\sqrt{f}} = -\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right),$$

$$\frac{\Delta P}{\rho} = \frac{fLv^2}{2D} \rightarrow v = \sqrt{\frac{2D\Delta P}{\rho f L}}.$$

Approach

- Let unknowns be v and $F \equiv 1/f^{1/2}$.
- Guess v_0 and F
- Solve for Re from the first equation
- Solve for F
- Solve the third equation for v
- Repeat

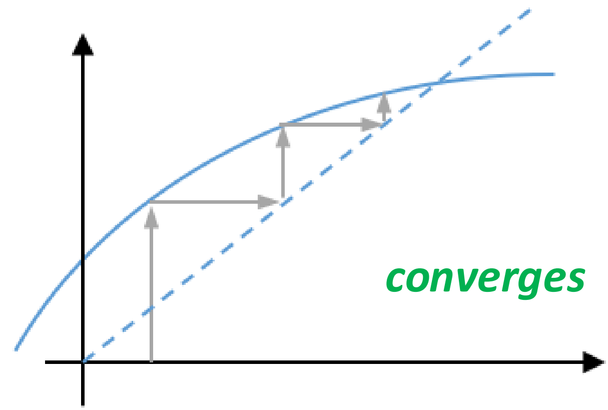
Convergence

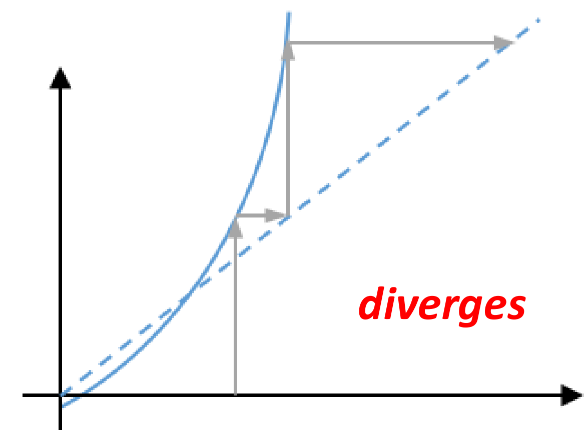
- For convergence, need the |slope|<1.
  - $x_{k+1}=g(x_k)$
  - At convergence, we have $x=g(x)$.
  - Subtract these two: $x_{k+1}-x = \epsilon_{k+1} = g(x_k)-g(x)$
  - Now, expand g as a Taylor series:

    $g(x)=g(x_k)+g'(\xi)(x-x_k)$, where $x \le \xi \le x_k$
  - Hence, $\epsilon_{k+1} = -g'(\xi)(x-x_k) = g'(\xi)\epsilon_k$
  - So, $\epsilon_{k+1}/\epsilon_k = g'(\xi)$.
  - For convergence, we need

    $|\epsilon_{k+1} / \epsilon_k| = |g'(\xi)|<1$.
  - At the solution, $|g'(x)|<1$.

$$\epsilon_{k+1} \propto \epsilon_k$$





Secant Method

- Like Regula Falsi, but always use the last two points

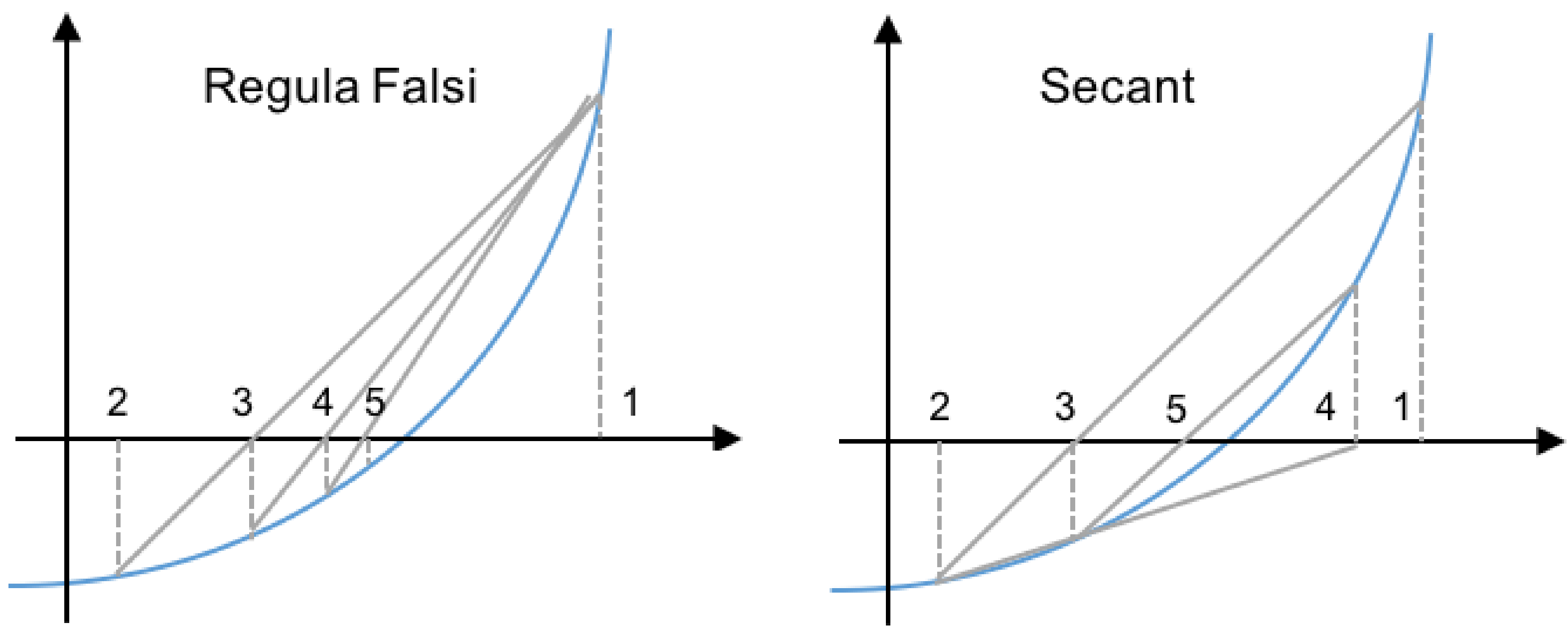
$$x_{k+1} = x_k - \frac{f(x_i)}{\hat{f}'_k}, \qquad \hat{f}_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

- Here, $\hat{f}_k$ is the slope of f at iteration k based on the last two iteration points.
- Might not bracket the root.
- But much faster convergence.
- This is an alternative to Newton's method, when we don't know or don't want to compute f'(x).
- Requires two starting points.

Convergence $\epsilon_{k+1} \propto \epsilon_k^{1.62}$

Numerical Recipes says that this is more efficient than Newton's method if the cost of evaluating f'(x)< 43% of evaluating f(x).

Secant Method



Newton’s Method

- Very popular
- Extends easily to multiple dimensions
- **Quadratic convergence:** $\epsilon_{k+1} \propto \epsilon_k^2$
- Approach:
  - Approximate the function as linear at the current value of x_k
  - Solve this linear equation for x_{k+1}
  - This x_{k+1} won't be the real answer because the real function is not linear But x_{k+1} will be an improvement
  - Repeat

Newton's Method

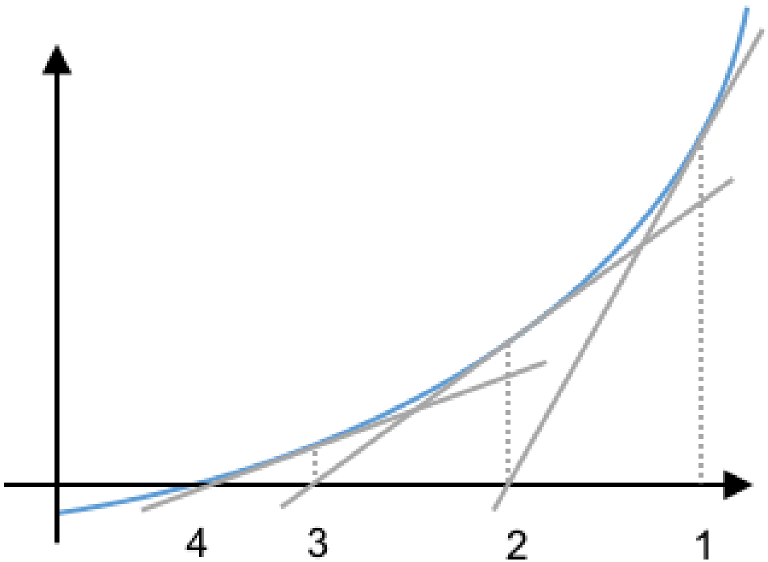
Linearize using a Taylor Series

$$f(x_{k+1}) = f(x_k) + f'(x_k)(x_{k+1} - x_k) + (\ldots)$$

Ignore terms (...)

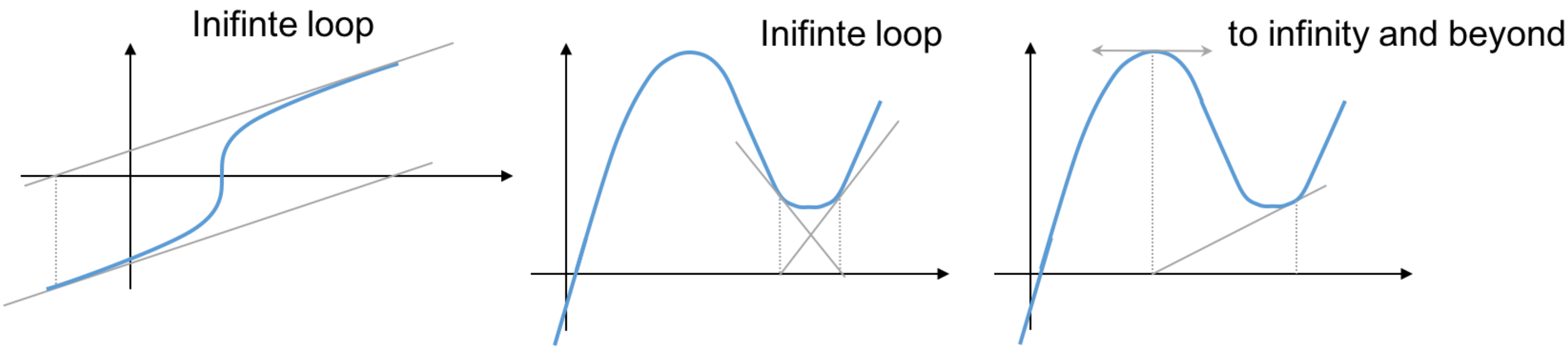
Set $f(x_{k+1})=0$ and solve for x_{k+1}

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$



Like Secant Method, but use $f'(x_k)$ instead of $\hat{f}'_k$

Problem Cases



Convergence

Recurrence relation

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

subtract x from both sides

$$\epsilon_{k+1} = \epsilon_k - \frac{f(x_k)}{f'(x_k)}$$

Taylor series

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2} f''(\xi)(x - x_k)^2$$

Let f(x) = 0, x_k - x = ε_k

$$f(x_k) = f'(x_k)\epsilon_k - \frac{1}{2} f''(\xi)\epsilon_k^2,$$

$$\frac{f(x_k)}{f'(x_k)} = \epsilon_k - \frac{1}{2}\frac{f''(\xi)}{f'(x_k)} \cdot \epsilon_k^2$$

Insert in green equation

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(x_k)} \cdot \epsilon_k^2.$$

Quadratic Convergence

- **Quadratic convergence**
  - Note, as $x_k \rightarrow x$, $\xi \rightarrow x$.
  - Takes a bit to get into the "quadratic" zone.
  - ...but once there, the number of significant digits roughly **doubles** at each iteration!
- **Example**
  - Solve $f(x)=x^2-\pi^2=0$ for x.
  - One solution is at $x = \pi =$ 3.141592653589793

```python
x = 1
print(f"x = {x:.15f}")
for k in range(6):
    x = x - (x**2 - np.pi**2)/(2*x)
    print(f"x = {x:.15f}")
```

```
x = 1.000000000000000
x = 5.434802200544679
x = 3.625401431921964
x = 3.173874724746142
x = 3.141756827069927
x = 3.141592657879261
x = 3.141592653589793
```

- k=0, x = 1.000000000000000
- k=1, x = 5.434802200544679
- k=2, x = **3.**625401431921964 → 1 digit
- k=3, x = **3.1**73874724746142 → 2 digits
- k=4, x = **3.141**756827069927 → 4 digits
- k=5, x = **3.14159265**7879261 → 8 digits
- k=6, x = **3.141592653589793** → 16 digits