# Class 19 - Conservation Eqns, Shvab - Zeldovich

- Detailed Derivations included

## Summary

Mass: $$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{v})$$

Species: $$\frac{\partial \rho Y_i}{\partial t} = -\nabla\cdot(\rho\vec{v}Y_i) - \nabla(\vec{j}_i) + \omega_i M_i$$

Momentum: $$\frac{\partial \rho\vec{v}}{\partial t} = -\nabla\cdot(\rho\vec{v}\vec{v}) - \nabla\cdot(\underline{\underline{\tau}}) - \nabla(P) + \rho\vec{g}$$

Energy: $$\frac{\partial \rho e}{\partial t} = -\nabla\cdot(\rho\vec{v}e) - \nabla\cdot(\vec{q}) - \nabla\cdot(\underline{\underline{\tau}}\cdot\vec{v}) - \nabla\cdot(P\vec{v}) + \rho\vec{g}\cdot\vec{v}$$

$$(\text{Accum}) \quad \underbrace{(\text{Convection}) \quad (\text{Diffusion})}_{-\nabla\cdot\,\text{flux}} \quad (\text{Source / work})$$

$-\nabla\cdot$ flux

↳ $\rho\vec{v} \;(=)\; \frac{kg}{m^2\cdot s}$ mass

$\rho\vec{v}Y_i \;(=)\; \frac{kg_i}{m^2\cdot s}$ species

$\rho\vec{v}\vec{v} \;(=)\; \frac{kg\cdot m/s}{m^2\cdot s}$ mom

$\rho\vec{v}e \;(=)\; \frac{J}{m^2\cdot s}$ energy

$$-\nabla\cdot\text{flux} \equiv (\text{in}) - (\text{out})$$


$$\int_{CV} -\nabla\cdot\vec{f}\,dV = -\int_{CS}\vec{f}\cdot\vec{n}\,dA = -\left[f_{out}\cdot A - f_{in}\cdot A\right]$$
$$= (f_{in}A) - (f_{out}A)$$

# Vars

Mass : $\rho$

Species : $Y_i$

Momentum : $\vec{v}$

Energy : $e$

$\Big\}$ Governing Eqns

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$P = \rho RT/M$ $\Big\}$ Equation of State.

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$j_i = j_i(T, P, Y_i)$

$\omega_i = \omega_i(T, P, Y_i)$

$\tau_i = \tau(T, P, Y_i, \vec{v})$

$q_i = q(T, P, Y_i, j_i)$

$\Big\}$ Constitutive Relations: $j_i, q, \tau$ (Source Term $\omega_i$)

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$T = T(Y_i, P, h)$

$h = e - \frac{1}{2}\vec{v}\cdot\vec{v} + \frac{P}{\rho}$

$\Big\}$ Thermodynamic Relations

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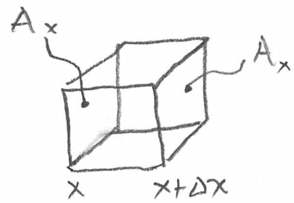
- with $c_p$, $D$, $\mu$, $\lambda$, $X_i$, $M_i$, etc.

## Derive

Species : Differential




Accum = in - out + gen

$$\frac{\partial m_i}{\partial t} = A_x(\rho Y_i V_i)_x - A_x(\rho Y_i V_i)_{x+\Delta x} + [Y, Z, \text{Dir's}] + \omega_i M_i \mathcal{V}_i$$

$$m_i = \rho Y_i \mathcal{V}_i = \rho Y_i \Delta x \Delta y \Delta z$$

$$A_x = \Delta y \Delta z$$

$$\div \ \Delta x \Delta y \Delta z$$

$$\frac{\partial \rho Y_i}{\partial t} = \underbrace{\frac{(\rho Y_i V_i)_x - (\rho Y_i V_i)_{x+\Delta x}}{\Delta x} + \frac{[Y, Z \text{ - Dirs}]}{\Delta Y, \Delta Z}} + \omega_i M_i$$

$\lim \Delta x \to 0$

$$-\frac{\partial \rho Y_i V}{\partial x} - \frac{\partial \rho Y_i V}{\partial y} - \frac{\partial \rho Y_i V_i}{\partial z} = -\nabla\cdot(\rho Y_i V_i)$$

$$V_i = V + V_i^D \longrightarrow \rho Y_i V_i = \rho Y_i V + \rho Y_i V_i^D$$
$$= \rho Y_i V + j_i$$

Species: integral.

RTT: $$\frac{dB}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho b \, d\mathcal{V} + \int_{CS} \rho b \vec{v}_R \cdot \vec{n} \, dA$$

- $B = m_i$, $b = B/m = Y_i$ • $\vec{V}_R = \vec{V}_i = \vec{v} + V_i^D$
- $$\frac{dB}{dt} = \frac{dm_i}{dt} = \int_{CV} M_i \omega_i \, d\mathcal{V}$$
- $$\int_{CV} M_i \omega_i \, d\mathcal{V} = \frac{\partial}{\partial t}\int_{CV} \rho Y_i \, d\mathcal{V} + \underbrace{\int_{CS} \rho Y_i \vec{V}_i \cdot \vec{n} \, dA}_{\int_{CV} \nabla\cdot(\rho Y_i \vec{V}_i)\, d\mathcal{V}}$$
- All terms $\int_{CV} \cdot \, d\mathcal{V}$

$$\longrightarrow \frac{\partial \rho Y_i}{\partial t} = -\nabla\cdot(\rho Y_i \vec{V}_i) + \omega_i M_i$$

$$\longrightarrow \frac{\partial \rho Y_i}{\partial t} = -\nabla\cdot(\rho Y_i \vec{V}) - \nabla\cdot j_i + \omega_i M_i$$

## Coordinates

$\nabla\cdot$

Cartesian: $$\nabla\cdot\vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Cylindrical: $$\nabla\cdot\vec{f} = \frac{1}{r}\frac{\partial r f_r}{\partial r} + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z}$$

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$\nabla$

Cartesian $$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial x}\vec{j} + \frac{\partial f}{\partial x}\vec{k}$$

Cylindrical $$\nabla f = \frac{\partial f}{\partial r}\vec{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{\theta} + \frac{\partial f}{\partial z}\vec{k}$$

- Do a lot with Cylindrical.
- See BSL App B, DGLaaT

## Shvab - Zeldovich Forms.

- Simplify Eqns for Flame analysis.

Energy

$$* \quad \rho \vec{V} \cdot \nabla h_s - \nabla \cdot (\rho D \nabla h_s) = - \sum h^o_{fi} \, \dot{m}_i'''$$

- $h_s = \int_{T_r}^{T} c_p \, dT$
- Turns (7.63 – 7.66.)
- Assumptions.
  - S.S.
  - $Le_i = \frac{\alpha}{D_i} = \frac{\lambda}{\rho c_p D_i} = 1 \quad (D_i = \lambda / \rho c_p)$
  - Fick's Law : $J_i = -\rho D_i \nabla Y_i$
  - No P.E. (gravity)
  - No Shaft work, Visc. Dissip.
  - No Radiation
  - No K.E. (compared to $\Delta H_{comb}$)
  - (No axial Diffusion B.L.A.)

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B.L. Approx : (Axisymmetric → Jets!)

- width small vs length
- $\frac{\partial}{\partial r} \gg \frac{\partial}{\partial x}$ (neglect axial Diffusion)
  - Mom
  - species
  - energy.
- $V_x \gg V_r$

# Details: Conservation Laws.

* Skip Section on Multicomponent Diffusion.

Friday = Movie: Jet fire.

Recap: Stoichiometry
Thermochemistry
Mass Transfer
Kinetics

Basic Coupling - Flow / Chemistry - 0D = PSR, PFR / Batch

Flames require Spatial Conservation 1D - 3D.

→ General Conservation Laws for Mass / Species, Momentum, Energy.

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- General Conservation Laws: Mass, Mom, Energy
  - Cartesian
  - Cylindrical
  - Spherical
- Schvab-Zeldovich forms (Simplify)
- Conserved-Scalar Equation (Mixture fraction)

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Derive Conservation laws

- Common: Balance on a C.V. Then Shrink to Differential
- Here: Start w/ R.T.T. Then apply Conservation Laws.
  - Integral form → Differential.

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- Reynold's Transport Theorem (RTT)

Conservation Laws: "Mass is not Created or Destroyed"

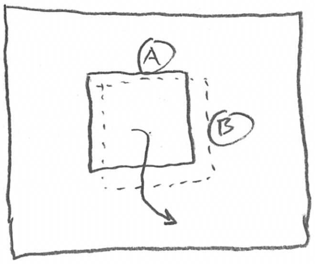
"$dU = dQ + dW$"

"$F = ma$"

These all apply to Lagrangian Systems (moving)

- we Define some Mass Then Describe Mass mom, energy Conservation of That moving mass.
- These are not Control Volume analyses!
- Conservation Laws are not written for fixed Eulerian Control Volumes

* The RTT Couples The Lagrangian System for which we have a Conservation law to The Eulerian Control Volumes Convenient for analysis




- Consider a <u>System of fixed mass</u>

(A)
  - It moves in space
  - It's Shape / Volume Deforms
  - <u>Lagrangian</u>

- Consider a Control Volume (CV) That is <u>fixed</u> is space / shape.

(B)
  - <u>Eulerian</u>

- At a given time, take The System and C.V. to overlap (but They only overlap for one instant)

- Let B be an extensive property of the system (Like Mass or Energy)
- Let $b = B/\text{mass}$

* RTT ≡ "Rate of Change of B in the lagrangian moving System = rate of Change of B in The fixed Eulerian C.V. + rate at which B leaves That C.V."

$$\underbrace{\frac{dB}{dt}}_{\text{Lagrangian System}} = \underbrace{\frac{d}{dt}\int_{CV} \rho b \, d\forall + \int_A \rho b \vec{V}\cdot\hat{n}\, dA}_{\text{Eulerian CV}}$$

; $\rho b \vec{V}$ is flux of B

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## <u>Mass Conservation</u>

- $B = \text{Mass}, \quad b = 1$
- Conservation law $\frac{dB}{dt} = \frac{dM}{dt} = 0$

$$\rightarrow 0 = \frac{d}{dt}\int_{CV} \rho\, d\forall + \int_A \rho \vec{V}\cdot\hat{n}\, dA$$

- Now pull $\frac{d}{dt}$ inside $\int$ (fixed C.V.)
- Use Gauss Divergence Theorem: $\int_A \vec{f} \cdot \vec{n}\, dA = \int_{CV} \nabla \cdot \vec{f}\, d\mathcal{V}$
- $0 = \int_{CV} \frac{\partial}{\partial t} \rho\, d\mathcal{V} + \int_{CV} \nabla \cdot (\rho \vec{v})\, d\mathcal{V}$
- $0 = \int_{CV} \left[ \frac{\partial}{\partial t} \rho + \nabla \cdot \rho \vec{v} \right] d\mathcal{V}$

$\rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0}$ Continuity equation.

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## Species Conservation

$B = M_i$ ; $b = Y_i$

Conservation Law; $\frac{dB}{dt} = \frac{dM_i}{dt} = \int_{CV} \underbrace{M_i w_i}_{} \, d\mathcal{V}$

$\int_{CV} M_i w_i\, d\mathcal{V} = \frac{d}{dt} \int \rho Y_i\, d\mathcal{V} + \int_A \rho Y_i \vec{v}_i \cdot \vec{n}\, dA$ $\frac{kg}{m^3 s}$

or rearrange, apply G.D.T.

$\rightarrow \frac{\partial \rho Y_i}{\partial t} + \nabla \cdot (\rho Y_i \vec{v}_i) = M_i w_i$

$v_i = v + v_i^D \rightarrow \rho Y_i v_i = \rho Y_i v + \rho Y_i v_i^D = \rho Y_i v + j_i$

$* \boxed{\frac{\partial \rho Y_i}{\partial t} + \nabla \cdot (\rho Y_i v) = - \nabla \cdot j + M_i w_i}$

$j = -\rho D \nabla Y_i$

$$\boxed{\frac{\partial \rho Y_i}{\partial t} + \nabla \cdot (\rho Y_i v) = \nabla \cdot (\rho D \nabla Y_i) + M_i w_i}$$

$= - \nabla \cdot j$

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## Momentum

$B = \text{Mom} = m\vec{v} \quad ; \quad b = \vec{v}$

Conservation Law ; $\dfrac{dm\vec{v}}{dt} = \sum \vec{F}_{ext}$ $\quad \left( F = ma = m\dfrac{dv}{dt} = \dfrac{dmv}{dt} \right)$

- Pressure and viscous forces

$$\sum \vec{F}_{ext} = -\int_A (P\underline{\underline{\delta}} + \underline{\underline{\tau}}) \cdot \vec{n}\, dA$$

$$-\int_A (P\underline{\underline{\delta}} + \underline{\underline{\tau}}) \cdot \vec{n}\, dA = \frac{d}{dt}\int_{CV} \rho\vec{v}\, d\forall + \int_A \rho\vec{v}\vec{v}\cdot\vec{n}\, dA.$$

Rearrange, use RTT

$$-\int_{CV} \nabla\cdot(P\underline{\underline{\delta}} + \underline{\underline{\tau}})\, d\forall = \frac{d}{dt}\int_{CV} \rho\vec{v}\, d\forall + \int_{CV} \nabla\cdot(\rho\vec{v}\vec{v})\, d\forall$$

$$\rightarrow \frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot\rho\vec{v}\vec{v} = -\nabla\cdot\underline{\underline{\tau}} - \nabla\cdot P\underline{\underline{\delta}}$$

$$\boxed{\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot\rho\vec{v}\vec{v} = -\nabla\cdot\underline{\underline{\tau}} - \nabla P} \quad + \rho\vec{g}$$

## Energy

- $B = E \quad ; \quad b = e = E/m \quad ; \quad e = \frac{1}{2}v^2 + u \quad ; \quad u = h - \frac{P}{\rho}$
- Conservation Law: $\dfrac{dE}{dt} = \dot{Q} + \dot{W} = -\int_A \vec{q}\cdot\vec{n}\, dA - \int_A \vec{F}\cdot\vec{v}\, dA + \int_{CV} \rho\vec{g}\cdot\vec{v}\, d\forall$
- $\vec{F} = (P\underline{\underline{\delta}} + \underline{\underline{\tau}})\cdot\vec{n}$ (dotting w/ $\vec{n}$ gives the force vector on the surface)

$$\frac{dE}{dt} = -\int_{C.V.} \nabla\cdot\vec{q} + \nabla\cdot(\underline{\underline{\tau}}\cdot\vec{v}) + \underbrace{\nabla\cdot(P\underline{\underline{\delta}}\cdot\vec{v})}_{\nabla\cdot(P\vec{v})}\, d\forall + \int_{CV} \rho\vec{g}\cdot\vec{v}\, d\forall$$

$$-\int_{CV} \nabla\cdot\vec{q} + \nabla\cdot(\underline{\underline{\tau}}\cdot\vec{v}) + \nabla\cdot P\vec{v}\, d\forall = \frac{d}{dt}\int_{CV} \rho e\, d\forall + \underbrace{\int_A \rho e\vec{v}\cdot\vec{n}\, dA}_{\int_{CV} \nabla\cdot(\rho e\vec{v})\, d\forall}$$

$$\Rightarrow \boxed{\frac{d\rho e}{dt} + \nabla\cdot(\rho e\vec{v}) = -\nabla\cdot q - \nabla\cdot(\tau\cdot\vec{v}) - \nabla\cdot P\vec{v} + \rho\vec{g}\cdot\vec{v}}$$

# Appendix B

# The Fluxes and the Equations of Change



## §B.1 NEWTON'S LAW OF VISCOSITY

$$[\boldsymbol{\tau} = -\mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger) + (\tfrac{2}{3}\mu - \kappa)(\nabla\cdot\mathbf{v})\boldsymbol{\delta}]$$

*Cartesian coordinates $(x, y, z)$:*

$$\tau_{xx} = -\mu\left[2\frac{\partial v_x}{\partial x}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla\cdot\mathbf{v}) \tag{B.1-1$^a$}$$

$$\tau_{yy} = -\mu\left[2\frac{\partial v_y}{\partial y}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla\cdot\mathbf{v}) \tag{B.1-2$^a$}$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla\cdot\mathbf{v}) \tag{B.1-3$^a$}$$

$$\tau_{xy} = \tau_{yx} = -\mu\left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right] \tag{B.1-4}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right] \tag{B.1-5}$$

$$\tau_{zx} = \tau_{xz} = -\mu\left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right] \tag{B.1-6}$$

in which

$$(\nabla\cdot\mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{B.1-7}$$

[a] When the fluid is assumed to have constant density, the term containing $(\nabla\cdot\mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity $\kappa$ is zero.

## §B.1 NEWTON'S LAW OF VISCOSITY (continued)

*Cylindrical coordinates $(r, \theta, z)$:*

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-8)}^a$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-9)}^a$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-10)}^a$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \qquad \text{(B.1-11)}$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right] \qquad \text{(B.1-12)}$$

$$\tau_{zr} = \tau_{rz} = -\mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right] \qquad \text{(B.1-13)}$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \qquad \text{(B.1-14)}$$

[a] When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity $\kappa$ is zero.

*Spherical coordinates $(r, \theta, \phi)$:*

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-15)}^a$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-16)}^a$$

$$\tau_{\phi\phi} = -\mu\left[2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta\cot\theta}{r}\right)\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \qquad \text{(B.1-17)}^a$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \qquad \text{(B.1-18)}$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] \qquad \text{(B.1-19)}$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] \qquad \text{(B.1-20)}$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \qquad \text{(B.1-21)}$$

[a] When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity $\kappa$ is zero.

## §B.2 FOURIER'S LAW OF HEAT CONDUCTION[a]

$$[\mathbf{q} = -k\nabla T]$$

*Cartesian coordinates $(x, y, z)$:*

$$q_x = -k \frac{\partial T}{\partial x} \tag{B.2-1}$$

$$q_y = -k \frac{\partial T}{\partial y} \tag{B.2-2}$$

$$q_z = -k \frac{\partial T}{\partial z} \tag{B.2-3}$$

*Cylindrical coordinates $(r, \theta, z)$:*

$$q_r = -k \frac{dT}{\partial r} \tag{B.2-4}$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta} \tag{B.2-5}$$

$$q_z = -k \frac{\partial T}{\partial z} \tag{B.2-6}$$

*Spherical coordinates $(r, \theta, \phi)$:*

$$q_r = -k \frac{\partial T}{\partial r} \tag{B.2-7}$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta} \tag{B.2-8}$$

$$q_\phi = -k \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \tag{B.2-9}$$

[a] For mixtures, the term $\Sigma_\alpha(\overline{H}_\alpha/M_\alpha)\mathbf{j}_\alpha$ must be added to $\mathbf{q}$ (see Eq. 19.3-3).

## §B.3 FICK'S (FIRST) LAW OF BINARY DIFFUSION[a]

$$[\mathbf{j}_A = -\rho \mathscr{D}_{AB} \nabla \omega_A]$$

*Cartesian coordinates $(x, y, z)$:*

$$j_{Ax} = -\rho \mathscr{D}_{AB} \frac{\partial \omega_A}{\partial x} \tag{B.3-1}$$

$$j_{Ay} = -\rho \mathscr{D}_{AB} \frac{\partial \omega_A}{\partial y} \tag{B.3-2}$$

$$j_{Az} = -\rho \mathscr{D}_{AB} \frac{\partial \omega_A}{\partial z} \tag{B.3-3}$$

*Cylindrical coordinates $(r, \theta, z)$:*

$$j_{Ar} = -\rho \mathscr{D}_{AB} \frac{\partial \omega_A}{\partial r} \tag{B.3-4}$$

$$j_{A\theta} = -\rho \mathscr{D}_{AB} \frac{1}{r} \frac{\partial \omega_A}{\partial \theta} \tag{B.3-5}$$

$$j_{Az} = -\rho \mathscr{D}_{AB} \frac{\partial \omega_A}{\partial z} \tag{B.3-6}$$

*Spherical coordinates $(r, \theta, \phi)$:*

$$j_{Ar} = -\rho \mathscr{D}_{AB} \frac{\partial \omega_A}{\partial r} \tag{B.3-7}$$

$$j_{A\theta} = -\rho \mathscr{D}_{AB} \frac{1}{r} \frac{\partial \omega_A}{\partial \theta} \tag{B.3-8}$$

$$j_{A\phi} = -\rho \mathscr{D}_{AB} \frac{1}{r \sin \theta} \frac{\partial \omega_A}{\partial \phi} \tag{B.3-9}$$

[a] To get the molar fluxes with respect to the molar average velocity, replace $\mathbf{j}_A$, $\rho$, and $\omega_A$ by $\mathbf{J}_A^*$, $c$, and $x_A$.

## §B.4 THE EQUATION OF CONTINUITY[a]

$$[\partial \rho / \partial t + (\nabla \cdot \rho \mathbf{v}) = 0]$$

*Cartesian coordinates $(x, y, z)$:*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{B.4-1}$$

*Cylindrical coordinates $(r, \theta, z)$:*

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{B.4-2}$$

*Spherical coordinates $(r, \theta, \phi)$:*

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0 \tag{B.4-3}$$

[a] When the fluid is assumed to have constant mass density $\rho$, the equation simplifies to $(\nabla \cdot \mathbf{v}) = 0$.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla$$

## §B.5 THE EQUATION OF MOTION IN TERMS OF $\boldsymbol{\tau}$

$$[\rho D\mathbf{v}/Dt = -\nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}]$$

*Cartesian coordinates $(x, y, z)$:*[a]

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} - \left[\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right] + \rho g_x \quad \text{(B.5-1)}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} - \left[\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right] + \rho g_y \quad \text{(B.5-2)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left[\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right] + \rho g_z \quad \text{(B.5-3)}$$

[a] These equations have been written without making the assumption that $\boldsymbol{\tau}$ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, $\tau_{xy}$ and $\tau_{yx}$ may be interchanged.

*Cylindrical coordinates $(r, \theta, z)$:*[b]

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right] + \rho g_r \quad \text{(B.5-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] + \rho g_\theta \quad \text{(B.5-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz}\right] + \rho g_z \quad \text{(B.5-6)}$$

[b] These equations have been written without making the assumption that $\boldsymbol{\tau}$ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, $\tau_{r\theta} - \tau_{\theta r} = 0$.

*Spherical coordinates $(r, \theta, \phi)$:*[c]

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$- \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta r}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\tau_{\phi r} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right] + \rho g_r \quad \text{(B.5-7)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$- \left[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\tau_{\phi\theta} + \frac{(\tau_{\theta r} - \tau_{r\theta}) - \tau_{\phi\phi}\cot\theta}{r}\right) + \rho g_\theta \quad \text{(B.5-8)}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot\theta}{r}\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$- \left[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3\tau_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\phi}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\tau_{\phi\phi} + \frac{(\tau_{\phi r} - \tau_{r\phi}) + \tau_{\phi\theta}\cot\theta}{r}\right] + \rho g_\phi \quad \text{(B.5-9)}$$

[c] These equations have been written without making the assumption that $\boldsymbol{\tau}$ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, $\tau_{r\theta} - \tau_{\theta r} = 0$.

$$\frac{\partial}{\partial t}(\rho \vec{v}) = -\nabla \cdot \rho \vec{v}\vec{v} - \nabla P - \nabla \cdot \underline{\underline{\tau}} + \rho \vec{g}$$

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial}{\partial x_j}(\rho v_j v_i) - \frac{\partial}{\partial x_i}P - \frac{\partial}{\partial x_j}\tau_{ji} + \rho g_i \qquad i\text{-Dir}$$

$$\rightarrow \frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x}(\rho u u) - \frac{\partial}{\partial y}(\rho v u) - \frac{\partial}{\partial z}(\rho w u) - \frac{\partial}{\partial x}P - \frac{\partial}{\partial x}\tau_{xx} - \frac{\partial}{\partial y}\tau_{yx} - \frac{\partial}{\partial z}\tau_{zx} + \rho g_x$$

## §B.6 EQUATION OF MOTION FOR A NEWTONIAN FLUID WITH CONSTANT $\rho$ AND $\mu$

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}]$$

*Cartesian coordinates $(x, y, z)$:*

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x \quad \text{(B.6-1)}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y \quad \text{(B.6-2)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(B.6-3)}$$

*Cylindrical coordinates $(r, \theta, z)$:*

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \quad \text{(B.6-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \quad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(B.6-6)}$$

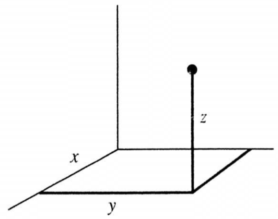
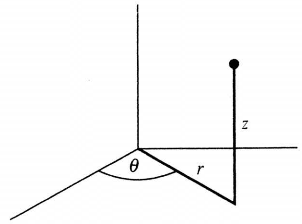
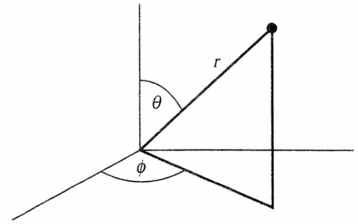
*Spherical coordinates $(r, \theta, \phi)$:*

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right] + \rho g_r \quad \text{(B.6-7)}^a$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta \quad \text{(B.6-8)}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\theta \quad \text{(B.6-9)}$$

[a] The quantity in the brackets in Eq. B.6-7 is *not* what one would expect from Eq. (M) for $[\nabla \cdot \nabla\mathbf{v}]$ in Table A.7-3, because we have added to Eq. (M) the expression for $(2/r)(\nabla \cdot \mathbf{v})$, which is zero for fluids with constant $\rho$. This gives a much simpler equation.

| | CARTESIAN | CYLINDRICAL | SPHERICAL |
|---|---|---|---|
| |   |   |   |
| DIVERGENCE<br>div **F**<br>$\nabla \cdot \mathbf{F}$ | $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ | $\frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$ | $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$ |
| GRADIENT<br>grad $f$<br>$\nabla f$ | $(\text{grad } f)_x = \frac{\partial f}{\partial x}$ | $(\text{grad } f)_r = \frac{\partial f}{\partial r}$ | $(\text{grad } f)_r = \frac{\partial f}{\partial r}$ |
| | $(\text{grad } f)_y = \frac{\partial f}{\partial y}$ | $(\text{grad } f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$ | $(\text{grad } f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$ |
| | $(\text{grad } f)_z = \frac{\partial f}{\partial z}$ | $(\text{grad } f)_z = \frac{\partial f}{\partial z}$ | $(\text{grad } f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$ |

# Details - Shvab-Zeldovich form,

- Last lecture we discussed the derivation of mass, species, momentum, energy using the Reynolds Transport Theorem
- These equations are powerful, physically accurate and provide many levels of description of reacting flows - combustion especially.
- The equations are general for gas-phase, but often more general than we need → simplify them.

* • An important simplification ~~ine~~ is called the Shvab - Zeldovich form of the energy equation.
  - Used in the description of premixed and nonpremixed flames to follow
  - Its not exact, so we don't use it in computer codes. But the simplifications make the math easier → solvable → theoretical analysis that is very useful for understanding flame behavior.

## Energy Eqn

- $$\frac{d\rho e}{dt} + \nabla\cdot(\rho e\vec{v}) = -\nabla\cdot q - \nabla\cdot(\tau\cdot\vec{v}) - \nabla\cdot p\vec{v} + \rho\vec{g}\cdot\vec{v}$$
  - No shaft work
  - $e = h - \frac{P}{\rho} + \frac{1}{2}v^2$
  - $q = -\lambda\nabla T + \sum h_i j_i + \dot{Q}''_{source}$

  } General ~~for~~ energy eqn.

- HW → (7.55)

$$\sum \dot{m}_i''' \frac{dh_i}{dx} + \frac{d}{dx}\left(-k\frac{dT}{dx}\right) + \dot{m}'' v_x \frac{dv_x}{dx} = -\sum h_i \dot{m}_i'''$$

  - SS
  - No P.E
  - 1-D Cartesian
  - No $\dot{Q}''_{source}$
  - No viscous dissipation

  • look at physical meaning of terms

- This is a more convenient form.

$= 7.55$ w/o K.E.

- HW → (7.67)

$$\dot{m}\, c_p \frac{dT}{dx} + \frac{d}{dx}\left(-\lambda\frac{dT}{dx}\right) + \underbrace{\sum \rho Y_i v_i^D}_{j_i} c_{p_i} \frac{dT}{dx} = -\sum h_i \dot{m}_i'''$$

## Shvab - Zeldovich

- Species mass fluxes, enthalpies eliminated
- enthalpy equation → looks just like the species equation

Start w/ Energy eqn.

→ SS, no KE, PE, Visc Diss,

$$\cancel{\frac{d\rho e}{dt}} + \underbrace{\nabla\cdot(\rho e\vec{v})}_{\nabla\cdot(\rho v h) - \cancel{\nabla\cdot(Pv)}} = -\nabla\cdot q - \cancel{\nabla\cdot(\tau\cdot\vec{v})} - \cancel{\nabla\cdot P\vec{v}} + \cancel{\rho \vec{g}\cdot\vec{v}}$$

w/o KE

$$\boxed{\nabla\cdot(\rho v h) = -\nabla\cdot q}$$

Turns 7.51

Convection + Diffusion = 0

## Heat Flux

$$\boxed{q = -\lambda\nabla T + \sum j_i h_i}$$

- Assume $j_i = -\rho D_i \nabla Y_i$
- Assume all $D_i$'s are equal → $D$
- Assume Unity Le ; $Le = 1 = \frac{\alpha}{D} = \frac{\lambda}{\rho C_p D} = 1 \rightarrow \alpha = D$

$$q = -\lambda\nabla T - \sum \rho D\, h_i \nabla Y_i$$

Now : $\nabla Y_i h_i = h_i \nabla Y_i + Y_i \nabla h_i \rightarrow h_i \nabla Y_i = \nabla Y_i h_i - Y_i \nabla h_i$

$$q = -\lambda\nabla T - \sum \rho D\, \overset{①}{\nabla Y_i h_i} + \sum \rho D\, \overset{②}{Y_i \nabla h_i}$$

① : • Pull $\sum_i$ inside $\nabla$ (This works cause $D_i = D$ (all same))

$\rightarrow \nabla \sum Y_i h_i = \nabla h$

② : • Also $\nabla h_i = C_{p_i} \nabla T \rightarrow \rho D \sum Y_i C_{p_i} \nabla T \rightarrow \rho D C_p \nabla T$

$$q = \cancel{-\lambda\nabla T} - \rho D\nabla h + \cancel{\rho D C_p \nabla T}$$

• but $\rho D C_p = \lambda$ by $Le = 1$

$$\boxed{q = -\rho D \nabla h}$$

nice !

Insert :

$$\boxed{\nabla\cdot(\rho V h) = \nabla\cdot\rho D\nabla h}$$ ≡ Turns 7.81, 7.82, 7.83

*

or

$$\rho\vec{V}\cdot\nabla h = \nabla\cdot\rho D\nabla h$$

• Note $\cancel{\frac{\partial\rho}{\partial t}}^{SS} + \nabla\cdot\rho\vec{V} = 0$ Continuity Mass

$\rightarrow \rho\vec{V}$ is Constant

• Enthalpy is a Conserved Scalar under our assumptions

• e.g. $\nabla\cdot(\rho\vec{V}h) - \nabla\cdot\rho D\nabla h = 0$ ; h is only transported

- There are no sources and no sinks.
- Its Transported as a whole, not as a Collection of Species enthalpies That all go "out of sync." This is Due to the Le = 1 assumption for all species.

• Now write in terms of $h_{sensible}$ : $h = h_f + h_{sens}$

→ replace the h w/ $h_s$, but get a Source term to account for Conversion of internal and sensible.

→ Shvab - Zeldovich.

$$\rho\vec{V}\cdot\nabla h_s + \rho\vec{V}\cdot\nabla h_f = \nabla\cdot\rho D\nabla h_s + \nabla\cdot\rho D\nabla h_f$$

$h_f = \sum Y_i h^\circ_{f_i}$ but $h^\circ_{f_i}$ is Constant → pull out of $\nabla$

$$\rho\vec{V}\cdot\nabla h_s - \nabla\cdot\rho D\nabla h_s = -(\rho\vec{V}\cdot\nabla h_f - \nabla\cdot\rho D\nabla h_f)$$

$$= -\left(\sum(\rho\vec{V} h_{f_i}\nabla Y_i - h_{f_i}\nabla\cdot\rho D\nabla Y_i\right)$$

$$= -\sum h_{f_i}\underbrace{(\rho\vec{V}\nabla Y_i - \nabla\cdot\rho D\nabla Y_i)}_{\dot{m}_i'''}$$

$$= -\sum h_{f_i}\dot{m}_i'''$$

$$\boxed{\rho\vec{V}\cdot\nabla h_s - \nabla\cdot\rho D\nabla h_s = -\sum h^\circ_{f_i}\dot{m}_i'''}$$ ≡ Shvab - Zeldovich Energy Eq.

• $h_s = \int_{T_{ref}}^{T} C_p\,dT$ Turns (7.64) (7.65) (7.66) (7.63)

Note: $h_s = \int_{T_{ref}}^{T} c_p\,dT \longrightarrow \frac{dh_s}{dx} = \frac{d}{dx}\int_{T_r}^{T} c_p\,dT = c_p\frac{dT}{dx}$

$$\nabla h_s \longrightarrow c_p \nabla T$$

$$\rho\vec{V} c_p \nabla T - \nabla\cdot\rho D c_p \nabla T = -\sum h^o_{fi}\,\dot{m}_i'''$$

- used in Premixed flames Turns (8.7a)

Compare energy and Species.

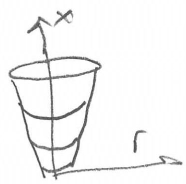
h: $\rho\vec{V}\cdot\nabla h_s - \nabla\cdot\rho D\nabla h_s = -\sum h^o_{fi}\,\dot{m}_i'''$

$Y_i$: $\rho\vec{V}\cdot\nabla Y_i - \nabla\cdot\rho D\nabla Y_i = \dot{m}_i'''$

(Convection) (Diffusion) (Source)

Note Signs — Creation of internal energy = + Creation of Sensible

Axisymmetric:




$$\nabla f \rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial r}$$

$$\nabla\cdot\vec{f} \rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{\partial}{\partial x} f_x$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r h_s) + \frac{\partial}{\partial x}(\rho V_x h_s) - \underbrace{\nabla\cdot\left(\rho D\frac{\partial h_s}{\partial x} + \rho D\frac{\partial h_s}{\partial r}\right)}_{= -\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho D\frac{\partial h_s}{\partial x} + r\rho D\frac{\partial h_s}{\partial r}\right) - \frac{\partial}{\partial x}\left(\rho D\frac{\partial h_s}{\partial x} + \rho D\frac{\partial h_s}{\partial r}\right)} = -\sum h^o_{fi}\,\dot{m}_i'''$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r h_s) + \frac{\partial}{\partial x}(\rho V_x h_s) - \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r\rho D\frac{\partial h_s}{\partial x}} + r\rho D\frac{\partial h_s}{\partial r}\right) - \frac{\partial}{\partial x}\left(\cancel{\rho D\frac{\partial h_s}{\partial x} + \rho D\frac{\partial h_s}{\partial r}}\right) = -\sum h^o_{fi}\,\dot{m}_i'''$$

Turns only keeps this term.
- Boundary-Layer approximation.