ChE 374–Lecture 6–Mass Balance

- Reynolds Transport Theorem: $\frac{B_{sys}}{dt} = \frac{d}{dt}\int_{CV}\rho b dV + \int_A \rho b\vec{v}\cdot\vec{n}dA$.

  $\rho b\vec{v}\cdot\vec{n}$ is the flux of B stuff through the control surface.

- Integral Mass Balance
  1. $B = M$, $b = B/M = 1$.
  2. Apply the conservation law: $dM/dt = 0$: mass is conserved.
  3. $\frac{d}{dt}\int_{CV}\rho dV + \int_A \rho\vec{v}\cdot\vec{n}dA = 0$.

- Cases
  1. Steady State
  2. Constant density
     * Liquids have approximately constant density
     * Gas density: ideal gas law: vary pressure (high speed flows or compressor), vary temperature (e.g., combustion), vary moles (e.g., combustion/reaction).
  3. Uniform inside C.V. and/or Uniform inlet/outlet flows over surface.
  4. Fixed or variable C.V. The control volume C.V. can move
     * $\vec{v} = \vec{v}_{sys} - \vec{v}_{cv}$.
     * Can pull the d/dt inside the integral.

  $\frac{dm}{dt} = \dot{m}_{in} - \dot{m}_{out}$

- Examples
  - Simple flow with change in area of inlet/outlet.
    * Fixed CV, SS, Constant ρ, Uniform Flows.
  - Filling a tank with inlet and outlet.
    * Constant ρ, Uniform Flows, not S.S., none-fixed C.V.
  - Nonuniform flow $\rightarrow$ compute the average velocity from a velocity profile.

- Differential Form of Mass Balance.
  - Get from the Integral form with a small/uniform property control volume and shrink to a point.
  - Get from the Gauss Divergence Theorem.

Class 6 - Integral Mass Balance.

R.T.T. : $\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, d\mathcal{V} + \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA$

Integral Mass Balance.

1. $B = M$ (kg)
   $b = \frac{B}{M} = 1$ $\longrightarrow$ $\frac{dM}{dt} = \frac{d}{dt}\int_{CV} \rho \, d\mathcal{V} + \int_{C.S} \rho \vec{V} \cdot \vec{n} \, dA$

2. Apply Conservation law $\frac{dM}{dt} = 0$ $\longrightarrow$ $\boxed{\frac{d}{dt}\int_{CV} \rho \, d\mathcal{V} + \int_{C.S} \rho \vec{V} \cdot \vec{n} \, dA = 0}$

   Thats it!

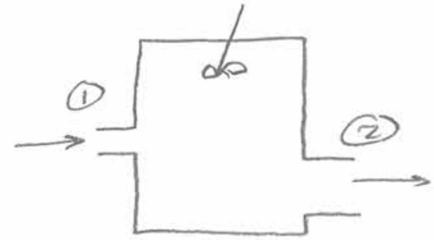
Cases

1. S.S. $\longrightarrow$ nothing Changes at a Given point
   $\frac{d}{dt}\int_{CV} \rho \, d\mathcal{V} = \frac{dM_{cv}}{dt} = 0$ at S.S.
   $\longrightarrow \int_A \rho \vec{V} \cdot \vec{n} \, dA = 0$

2. Const ρ $\longrightarrow$ $\div$ through by ρ, $\int_{CV} d\mathcal{V} = \mathcal{V}$
   $\longrightarrow \frac{d\mathcal{V}_{cv}}{dt} + \int_{C.S} \vec{V} \cdot \vec{n} \, dA = 0$
   if The C.V. is fixed Then $\int_{C.S} \vec{V} \cdot \vec{n} \, dA = 0$

- Liq: $\rho \sim$ const.
- Gases $\rho = MP/RT \longrightarrow P, T, M$ vary
  - High Speed flow $\rightarrow$ high ΔP
  - Reactions $\longrightarrow \Delta T$, Δ moles $\rightarrow \Delta M$
  - For low speed, $\sim$ Isothermal, $\rho_{gas} \approx$ Const.

3 Uniform Props inside C.V.
Uniform inlets, outlets



$$\frac{d}{dt}\int_{cv} \rho \, d\mathcal{V} + \int_{cs} \rho \vec{V}\cdot\vec{n}\, dA = 0$$

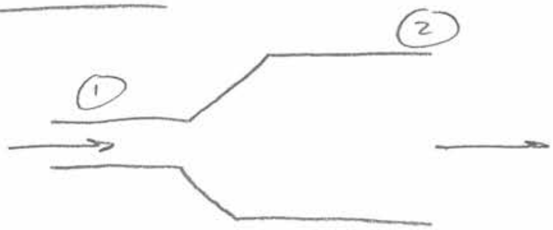
$$\frac{d\rho\mathcal{V}}{dt} + (\rho V A)_{out} - \underbrace{(\rho V A)_{in}}_{\dot{m}} = 0$$

$\vec{V}\cdot\vec{n}$ @ 1 is neg
$\vec{V}\cdot\vec{n}$ @ 2 is pos

$$\frac{dM_{cv}}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

$\mathcal{V}$, ρ can change in time.

Example 1



water, S.S.
Q: Fixed C.V. $\rightarrow$ Y
Q: SS $\rightarrow$ Y
Q: Const ρ $\rightarrow$ Y
Q: Unif in/out $\rightarrow$ Y.

S.S. $\rightarrow$ $\int_{cs} \rho \vec{V}\cdot\vec{n}\, dA = 0$

Const ρ $\rightarrow$ $\int_{cs} \vec{V}\cdot\vec{n}\, dA = 0$

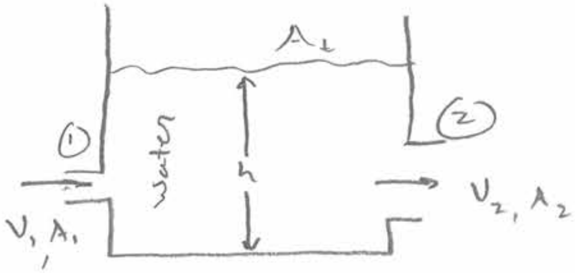
Uniform $\rightarrow$ $V_2 A_2 - V_1 A_1 = 0$

or $V_2 A_2 = V_1 A_1$
or $(\rho A V)_1 = (\rho A V)_2$
$\rightarrow$ $\dot{m}_{in} = \dot{m}_{out}$
or $\dot{\mathcal{V}}_{in} = \dot{\mathcal{V}}_{out}$.

$$\boxed{V_2 = V_1 \frac{A_1}{A_2}}$$

Volume flow in = Volume flow out.

Example 2 : Given V_1 V_2 A_1 A_2 A_t
Find $h(t)$



Q: Fixed C.V. ? → N
Take C.V. = water in Tank

Q: Const ρ ? → Y

Q: Uniform ? → Y

Q: SS ? → N

$$\text{Const } \rho \rightarrow \frac{d\mathcal{V}}{dt} + \int_A \vec{v}\cdot\vec{n}\,dA = 0$$

$$\text{Unif} \rightarrow \frac{d\mathcal{V}}{dt} + V_2 A_2 - V_1 A_1 = 0$$

$$\mathcal{V} = A_t h \quad , \longrightarrow \div \text{ by } A_t \text{ , rearrange}$$

$$\frac{dh}{dt} = \underbrace{\left(\frac{V_1 A_1 - V_2 A_2}{A_t}\right)}_{\text{Const } C}$$

$$\text{Solve} \rightarrow h = h_0 + \left(\frac{V_1 A_1 - V_2 A_2}{A_t}\right) t$$

Suppose $V = V(h)$?
$V = V(t)$? insert in ODE then solve ODE

* Always try to align the C.V., C.S. to be $\perp$ to the Velocities

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Example : Nonuniform Flow

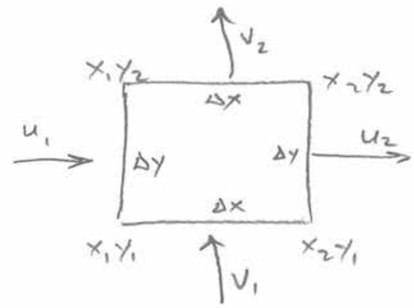
$$\dot{\mathcal{V}} = VA \rightarrow d\dot{\mathcal{V}} = v\,dA \rightarrow \dot{\mathcal{V}} = \int v\,dA$$

$$\overline{V} = \frac{1}{A}\int v\,dA \longrightarrow \dot{\mathcal{V}} = \int v\,dA = A\overline{V}$$

Given a velocity profile, Find $\overline{V} = \frac{1}{A}\int v(x)\,dA$

- Plug in Given $V(x)$, Set dA, such as $dA = w\,dx$
- Integrate.

Differential Mass Balance.



Fixed C.V., uniform

$$\frac{d}{dt}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}\rho\,\vec{V}\cdot\vec{n}\, dA = 0$$

$$\mathcal{V}\frac{d\rho}{dt} + \rho_{x2}u_2\Delta y - \rho_{x1}u_1\Delta y + \rho_{y2}v_2\Delta x - \rho_{y1}v_1\Delta x = 0$$

$\downarrow$

$\Delta x\Delta y$ • here Areas are $\Delta x, \Delta y$ for a unit Δz in the Page.

$$\div\ \Delta x, \Delta y \longrightarrow \frac{d\rho}{dt} + \frac{\rho_{x2}u_2 - \rho_{x1}u_1}{\Delta x} + \frac{\rho_{y2}u_2 - \rho_{y1}v_1}{\Delta y} = 0$$

$$\lim_{\Delta x, \Delta y \to 0} \longrightarrow \frac{\partial\rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\longrightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0}$$

Optional

Gauss Divergence Theorem (GDT)

$$\int_A \vec{V}\cdot\vec{n}\, dA = \int_{CV}\nabla\cdot\vec{V}\, d\mathcal{V} \qquad \equiv \text{GDT}$$

Then

$$\frac{d}{dt}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}\rho\,\vec{V}\cdot\vec{n}\, dA = 0$$

$$\longrightarrow \frac{d}{dt}\int_{CV}\rho\, d\mathcal{V} + \int_{CV}\nabla\cdot(\rho\vec{V})\, d\mathcal{V} = 0$$

$$\longrightarrow \int_{CV}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right) d\mathcal{V} = 0$$

$$\longrightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0}$$