ChE 374–Lecture 5–Math for Property Balances

- Math Review
  - Scalar Field: $s(x, y, z)$
    - One value at each point
  - Vector Field: $\vec{v}(x, y, z) = u(x, y, z)\vec{i} + v(x, y, z)\vec{j} + w(x, y, z)\vec{k}$
    - Three values at each point.
  - Tensor Field:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}.$$

    - Nine values at each point. Two directions at each point.
    - Think of a general force vector on a general surface. We represent the surface as projections onto a cube in x,y,z, and the force on each surface has three components, giving 9 total components.
    - τ_{yz} is the shear stress on the y plane in the z direction.
    - We've already seen τ_{xy} in defining μ.
  - Dot product
  - Flux (quantity per unit time, per unit area). Differential flux: $-q\vec{v} \cdot \vec{n}dA$ where q is stuff per volume.
- Lagrangian and Eulerian Reference Frames.
  - Goal: Write balance equations to describe and solve problems.
    - Conservation laws = mass, momentum, energy, written for a System of Fixed
    - The practical equations don't concern fluid elements with fixed masses, but fixed regions, so connect these two.
  - Lagrangian: Motion of a fixed mass (system).
    - Conservation laws written in the Lagrangian frame.
  - Eulerian: Some fixed control volume through which fluid moves.
- Material Derivative (differential)

  $\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{v} \cdot \nabla \phi.$
    - (the rate of change of the moving point) = (the rate of change at the fixed point) + (the rate of change due to motion in the nonuniform field). mass.
- Reynolds Transport Theorem, RTT (integral)
  - Consider an extensive quantity B (like total mass or energy) of a system, and b is just B per unit mass (intensive).

    $\frac{dB_{syst}}{dt} = \frac{d}{dt}\int_{CV} \rho b d(CV) + \int_A \rho \vec{v} \cdot \vec{n} dA.$
    - (Lagrangian) = (Eulerian).
    - (the rate of change of B in the system) = (the rate of change of B in the fixed C.V.) + (the rate that B leaves the C.V.).
      - At a given instant, the stuff entering the C.V. is not part of the system, the stuff leaving the C.V. is.
    - Note: $\vec{v}$ is the system velocity; if the C.V. moves, then $\vec{v}$ is the velocity of the system relative to the C.V.

Class 5 - Math for Property Balances

- Move from Statics where $\Sigma F = 0$ is basic relation, to fluids in motion
- Need Eqns to balance Mass, Momentum, Energy
- Outline
  - Math review
  - Lagrangian, Eulerian Frames
  - Material Derivative
  - Reynolds Transport Theorem.

Math review

Scalar field $\longrightarrow$ 1 number, $s(x, y, z, t)$, like Pressure

Vector field $\longrightarrow$ 3 numbers, $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$

with $u = u(x, y, z, t)$, etc.

e.g. velocity, or force, or acceleration.

3 oriented Scalar values at each location

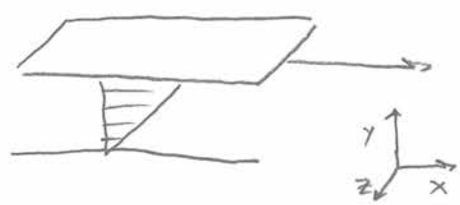
Tensor field $\longrightarrow$ 9 numbers

- Scalar $\longrightarrow$ 0 Directions
- Vector $\longrightarrow$ 1 Direction
- tensor $\longrightarrow$ 2 Directions

- Stress tensor $\underline{\underline{\tau}} =$

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{matrix} \tau_{xx}\vec{i}\vec{i} + \tau_{xy}\vec{i}\vec{j} + \tau_{xz}\vec{i}\vec{k} + \\ \tau_{yx}\vec{j}\vec{i} + \tau_{yy}\vec{j}\vec{j} + \tau_{yz}\vec{j}\vec{k} + \\ \tau_{zx}\vec{k}\vec{i} + \tau_{zy}\vec{k}\vec{j} + \tau_{zz}\vec{k}\vec{k} \end{matrix}$$

- We've seen τ_{yx}:

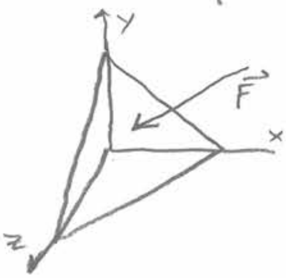


τ_{yx} : y - Plane
x - Direction

$\longrightarrow$

But There is also Shear stress in z-Dir if Plate moves in z too $\swarrow$ $\rightsquigarrow \tau_{yz}$

- $\tau_{yx} = \mu \frac{du}{dy} \rightarrow$ Velocity Gradient.
- 3 components of vel in 3 Directions $\rightarrow$ 9 components
- In Terms of an arbitrary Surface
  - Surface projects to our 3 Cartesian Planes
  - The Force $\vec{F}$ on The Surf. Has 3 components $\rightarrow 3 \times 3 = 9$

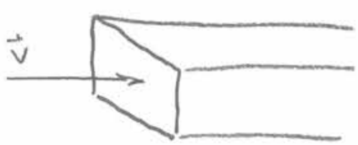


Dot Product

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = \text{Scalar}$$
$$= |a||b| \cos\theta$$
$$\vec{a} \cdot \vec{n} = |a| \cos\theta$$ Projects $\vec{a}$ onto Direction $\vec{n}$

Flux

Flux is quantity per area per time ; e.g. $\frac{kg}{m^2 \cdot s}$

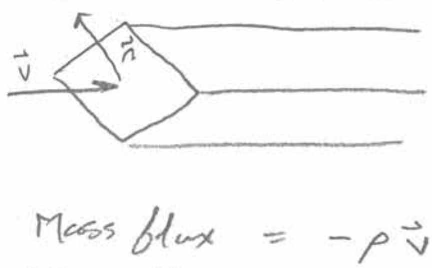


$\rho V \rightarrow kg/m^2 \cdot s$

$\rho A V \rightarrow kg/s$

See $\boxed{\text{Slide}}$

If $\vec{V}$ is not $\perp$ to Surface:



$\vec{n}$ is unit normal $\perp$ to Surface, Pointing <u>out</u> of Surf.

Mass flux $= -\rho \vec{V} \cdot \vec{n}$

Mass flow $= -\rho \vec{V} \cdot \vec{n} A$

Energy flux $= -\rho h \vec{V} \cdot \vec{n}$ $\quad (=) \; \frac{\cancel{kg}}{m^3} \cdot \frac{J}{\cancel{kg}} \cdot \frac{m}{s} = \frac{J}{m^2 \cdot s}$

Energy flow $= -\rho h \vec{V} \cdot \vec{n} A$

Lagrangian, Eulerian Frames

See $\boxed{\text{Slides}}$

Connect Lagrangian / Eulerian → Differential / Integral

Material Derivative - Differential

Consider property ϕ : Like P or velocity component

$$\phi = \phi(x, y, t)$$

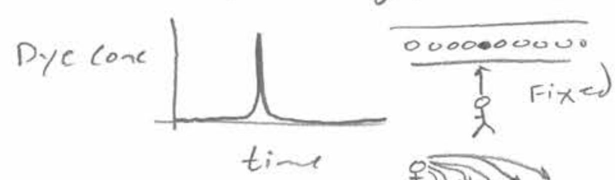
Eulerian : $\left(\frac{\partial \phi}{\partial t}\right)_{x,y}$ or $\left(\frac{\partial \phi}{\partial x}\right)_{y,t}$ etc.

→ change in ϕ in time Direction without changing x, y

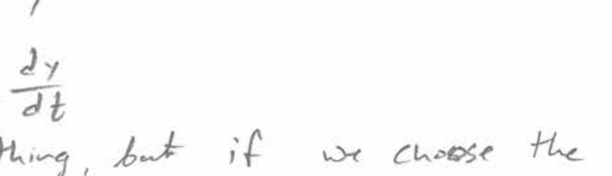
→ Change in ϕ in x - Dir without changing y, or t

Lagrangian : Variation of <u>system</u> as it moves

- A Drop of dye in a river: Follow The Drop and watch how it changes its Concentration.
- In The Eulerian we'd watch a point (fixed spot) in The river :



- Lagrangian



$$\phi = \phi(t, x, y)$$

$$d\phi = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \qquad \text{Total Derivative.}$$

$\div dt$ $$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt}$$

$\frac{dx}{dt}, \frac{dy}{dt}$ can be anything, but if we choose the

$$\frac{dx}{dt} = u \text{ of fluid and } \frac{dy}{dt} = v \text{ of fluid}$$

→

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y}$$

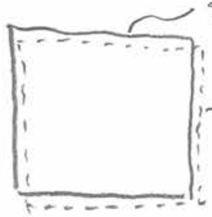
or $$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{V}\cdot\nabla\phi \longrightarrow \boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\nabla}$$

Substantial or material Derivative.

* (The rate of change of ϕ of the moving point) =
(The rate of change of ϕ in a given spot) +
(The change in ϕ with position ($\nabla\phi$) times the change in position with time ($\vec{V}$); this product gives a rate of change of ϕ due to motion in the nonuniform $\phi(x,y)$ field)

Reynolds Transport Theorem - Integral.

System



Control Volume C.V. with Control Surface C.S.

- System of Fixed mass
- moves in space, shape, volume Deforms

- Control Volume
- Fixed in size, shape, location

- At a given time, the sys, C.V. overlap (but only at that instant since the system is moving, and the C.V. is fixed)

- Let B be an extensive quantity like mass (kg), energy (J)
- Let $b = B/\text{mass}$ is intensive.

- RTT:
$$\underbrace{\frac{dB_{sys}}{dt}}_{\text{Lag.}} = \underbrace{\underbrace{\frac{d}{dt}\int_{CV}\rho b\, dV}_{\text{Change in C.V.}} + \underbrace{\int_{CS}\rho\vec{V}\cdot\vec{n}\, dA}_{\text{Flow Through Surf}}}_{\text{Eul.}}$$

Again: $$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, d\mathcal{V} + \int_{cs} \rho \vec{V}\cdot\vec{n}\, dA$$

(Rate of Change of B in Sys (which moves)) =
(Rate of Change of B in The Fixed C.V.) +
(Rate That B Crosses out of the C.S.)

or: Generation = Accum + out − in

Rearrange to:

Accum = in − out + Generation.

- Stuff entering C.V. is not our System, stuff leaving is
- These are rates. At a given time instant, The Sys, C.V. overlap
- $\vec{V}$ is The system velocity
- If C.V. moves or Deforms, Then $\vec{V} = \vec{V}_{rel} = \vec{V}_{sys} - \vec{V}_{cs}$
- Ignore The Book's Caveat on

$$\frac{dB}{dt} = \frac{d}{dt}\int_{CV} \rho b \, d\mathcal{V} + \int_{c.s} \rho \vec{V}_R\cdot\vec{n}\, dA$$
$$= \int_{CV} \frac{\partial}{\partial t} \rho b \, d\mathcal{V} + \int_{c.s.} \rho \vec{V}_{abs}\cdot\vec{n}\, dA$$

(its not physically intuitive, and just admits The Leibniz Theorem)

- Like $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\nabla$ in Substantial Derivative

But, ↗ Derivative is Translation only, no Deformation

→ $\int$ Material Derivative ≠ RTT

Chemical Engineering 374

Fluid Mechanics
Fall 2011

Math Tools for Fluids

BYU

1

Flux

2

- Flux = quantity per unit area per unit time
  - Heat flux : J/m^2*s
  - Mass flux: kg/m^2*s
  - Momentum flux: kg*m/s*m^2*s = kg/ms^2
- ρv = kg/m^3 * m/s = kg/m^2*s = mass flux
- Then the flux of any quantity per unit mass (q) is
  - $\rho q\mathbf{v}$
    - q = h → J/m^2*s Heat flux
    - q=1 → kg/m^2*s Mass flux
    - q=v → kg/s*m^2*s = kg/m*s^2 Momentum flux
- -r*q*$\mathbf{v}\cdot\mathbf{n}$*A is the rate of quantity through surface A

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Lagrangian/Eulerian

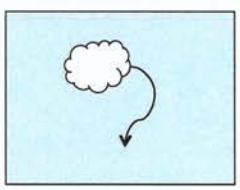
3

- As we go to moving fluids (not fluid statics)…
- GOAL: Write balance equations to describe and solve problems
- Conservation Laws:
  - Mass "The mass of an object is converved (not created/destroyed)"
  - Momentum "Acceleration of an object = net force on the object"
  - Energy "Energy of a given mass is conserved"
- All these laws are written in terms of some object, or some fixed mass
- In Engineering, we don't normally care about some object, but some fixed region in space.
  - We care about the pump, not the "piece" of fluid flowing through it.
  - While the mass of a "piece" of fluid is conserved, the mass inside a pump can change.
  - The conservation law is written for the piece of fluid, NOT the pump.
- So how do we get a conservation law for a piece of fluid in terms of a region of space?
  - Two frames of reference: **Lagrangian** (piece of fluid) ←→ **Eulerian** (region of space)

Lagrangian/Eulerian

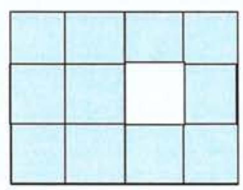
4

Lagrangian



- Motion of system of **fixed mass**
- CONSERVATION LAWS
- Fluid elements move around and deform

Eulerian



- Some **fixed control volume**
- CONVENIENT FOR ENGINEERING
- Don't care about fluid elements
- Want pressure and velocity fields at a point.
  - Pressure on a wing
  - Drag on a car
  - Not the pressure of a chunk of fluid as it moves along