ChE 374–Lecture 37–Compressible Flows

- Compressible flows occur at high speeds (B.E. $\Delta P = 1/2\rho\Delta v^2$.)
  - Gas can convert significant internal energy to kinetic energy.
  - Density decreases, temperature decreases.
  - Mach number: $\mathcal{M} = v/c$, where c is the *local* speed of sound.
  - Compressible flow for $\mathcal{M} > 0.3 \rightarrow$ density ratio $\approx 0.95 \rightarrow 5\%$ difference.
    * Most scalings go with $\mathcal{M}^2$.
- Sound Speed
  - Follow a pressure pulse (frame of the pulse).
    * Mass balance (ρAv) and momentum balance (pressure force, flow in, flow out) give (for ideal gases):

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} = c = \sqrt{\frac{kRT}{M}},$$

      where $k = C_P/C_V$, and M is the mean molecular weight.
  - Values for air:
    * $k = 1.4$
    * Air at 25 C = 77 F: $c = 346\,m/s = 774\,mph$
    * Air at 2000 K = 3140 F: $c = 896\,m/s = 2000\,mph$
      · (heavier gases give slower speed, and higher temperatures give higher speed).
  - Water at 15 C $c = 1490\,m/s = 3333\,mph.$
  - Steel at 15 C $c = 5060\,m/s = 11318\,mph.$
- Analysis: SS, no friction, 1D, Ideal Gas
  - Energy balance from a reservoir into some pipe/valve/etc.
    * No heat, no shaft work, no gravity effect, $P/\rho + u = h$, and $dh = C_p dT$.
      Use definition of k, and c given above to obtain:

$$\frac{T_r}{T_1} = \frac{\mathcal{M}_1^2(k-1)}{2} + 1,$$

      where r denotes the reservoir (no velocity), and subscript 1 is the point of interest.
  - For ideal gases: $P_r/P_1 = (T_r/T_1)^{k/(k-1)}$, $\rho_r/\rho_1 = (T_r/T_1)^{1/(k-1)}$. So:

$$\frac{P_r}{P_1} = \left(\mathcal{M}^2\frac{k-1}{2} + 1\right)^{k/(k-1)}$$

$$\frac{\rho_r}{\rho_1} = \left(\mathcal{M}^2\frac{k-1}{2} + 1\right)^{1/(k-1)}$$

- Flow in nozzles.
  - For incompressible flow, the density is constant and have $\dot{m} = \rho Av$.
  - For compressible flow, the density drops. For $\mathcal{M} < 1$, the density drops slower than velocity increases for a given change in area and we need a converging nozzle as usual to increase the velocity through the nozzle. For $\mathcal{M} > 1$, the density drops faster than velocity increases for a given change in area and we need a *diverging* nozzel to increase the velocity through the nozzle. Hence the converging/diverging nozzles on rockets.
- Choked Flow: Pressure ratios below 0.53, result in choked flow in valves, etc. For choked flow, there is a maximum flow rate, independent of what is done downstream.
  - Most control valves, safety valves, bike tire valves are choked.
  - **YOU MUST KNOW AND UNDERSTAND THIS CONCEPT**

Class 37 - Compressible Flows

Previously: Const ρ.

Now, ρ varies

- High Speed flow, large ΔP, ΔV, ΔT, etc.
- Gas converts internal energy to kinetic energy
    - T Drops ; ρ Drops
    - V > Bernoulli eqn where ρ is const.
- Mach number $M = V/c$; c is the local value since c varies with temperature.
- Compressible for $M > 0.3 \rightarrow \frac{\rho}{\rho} \sim 0.95 \rightarrow 5\%$
- Most Scalings Go with M^2

Sound Speed

Follow a Pressure Pulse

- Ride the pulse → Flow in/out of C.V. moving at Sound Speed : c

Mass Balance: $\dot{m} = \dot{m} \rightarrow \rho A V = (\rho + d\rho) A (V + dV)$

$$\rightarrow \cancel{\rho V} = \cancel{\rho V} + \rho dV + V d\rho + \cancel{d\rho dV}$$

ignore

$$\rightarrow \boxed{\rho dV = -V d\rho}$$

Momentum Balance: SS, only Pressure forces

$$AP - A(P + dP) = \dot{m}(V + dV) - \dot{m}V$$

$\dot{m} = \rho A V$, A cancels.

$$-dP = \rho V dV \rightarrow \boxed{\rho dV = -\frac{dP}{V}}$$

insert

Insert $\rho dV = -V d\rho$ from Mass Bal

$$\frac{dP}{d\rho} = V^2 = c^2 \longrightarrow c = \sqrt{\frac{dP}{d\rho}} \longrightarrow$$

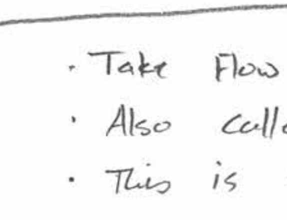
- $P = \frac{\rho R T}{M}$; $\frac{dP}{d\rho} \rightarrow \left(\frac{\partial P}{\partial \rho}\right)_T$ or $\left(\frac{\partial P}{\partial \rho}\right)_s$
- Sound waves compress → no heat Transfer
  → isentropic, But not isothermal
- $c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} = \boxed{\sqrt{\frac{kRT}{M}} = c}$
- $k = C_P / C_V$
- $k = 1.4$ for Air

	c (m/s)	c (mph)
Air @ 298 K	346	774
Air @ 2000 K	896	2000
H_2O @ 15°C	1490	3333
Steel @ 15°C	5060	11318

- Note: $\left(\frac{\partial P}{\partial \rho}\right)_T = \frac{RT}{M} \longrightarrow$ off by $\sqrt{k}$ = 18% or 84%

Analysis

- Take Flow from Some Large Reservoir, $V=0$.
- Also called Stagnation Condition Corresponding to a Given flow.
- This is a Convenient Reference State.
- Assume: S.S., 1-D, Ideal Gas, no friction, ignore Gravity
- Energy Balance: $\cancel{\dot{Q}} + \cancel{\dot{W}_s} = \dot{m}\left(\underbrace{\frac{P}{\rho} + u}_{h} + \cancel{\frac{V^2}{2}} + \cancel{gz}\right)_R - \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + \cancel{gz}\right)_1$



$$\longrightarrow V_1^2 = 2(h_r - h_1)$$
$$= 2C_p(T_r - T_1)$$

Now: $C_p = C_v + \frac{R}{M}$

$$k = C_p/C_v \longrightarrow C_v = C_p/k$$

$$C_p = \frac{C_p}{k} + \frac{R}{M} \longrightarrow C_p = \frac{Rk}{M(k-1)} \longrightarrow \text{insert above}$$

$$V_1^2 = \frac{2Rk}{M(k-1)}(T_r - T_1)$$

use $c^2 = \frac{kR}{M}T$

$$\frac{V_1^2}{c_1^2} = M^2 = \frac{2}{k-1}\left(\frac{T_r}{T_1} - 1\right)$$

$$\rightarrow \boxed{\frac{T_r}{T_1} = \frac{M^2(k-1)}{2} + 1}$$

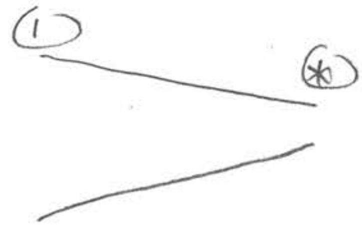
For ideal Gases (see Thermo Class next Semester ü)

$$\frac{P_R}{P_1} = \left(\frac{T_R}{T_1}\right)^{k/(k-1)} \quad ; \quad \frac{\rho_R}{\rho_1} = \left(\frac{T_R}{T_1}\right)^{\frac{1}{k-1}}$$

Then, using $\frac{T_R}{T_1}$ above,

$$\boxed{\begin{aligned} \frac{P_R}{P_1} &= \left(M^2\frac{k-1}{2} + 1\right)^{\frac{k}{k-1}} \\ \frac{\rho_R}{\rho_1} &= \left(M^2\frac{k-1}{2} + 1\right)^{\frac{1}{k-1}} \end{aligned}}$$

Nozzles.



$\dot{m} = \rho A V = \text{const}$

As A Decreases V increases,

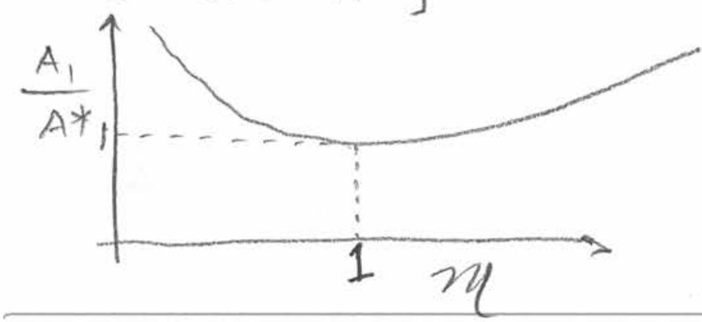
Eventually $V = c$ at (*)

$$\frac{A_1}{A^*} = \frac{\rho^* V^*}{\rho_1 V_1} \quad ; \quad \text{insert } \rho = \rho_R \left[M^2 \frac{k-1}{2} + 1 \right]^{-\frac{1}{k-1}}$$

(note that our Reference Reservoir Condition ρ_R cancels). Note M at (*) = 1

$$* \quad \frac{A_1}{A^*} = \frac{1}{M_1} \left[\frac{M_1^2 (k-1)/2 + 1}{(k-1)/2 + 1} \right]^{(k+1)/2(k-1)}$$

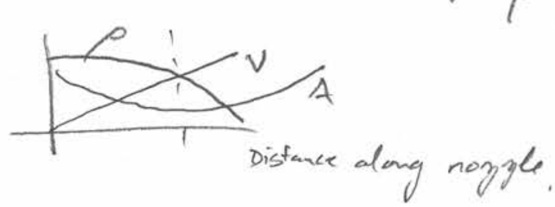
Plot →



For $M < 1$, As velocity increases, A Decreases.

For $M > 1$, As velocity increases, A increases!

- Because ρ is Decreasing Faster Than A is Decreasing → V increases to keep $\dot{m} = \rho A V$ Constant!



Choked Flow.

Consider a Nozzle

- Start with high P in a Reservoir and Drop the pressure outside.
- Velocity increases in nozzle till hit Sonic Velocity, $M = 1$
- At that Point, further Reduction cannot be communicated upstream
- If I shout into the wind: If the wind speed = sound speed, then I cannot communicate; my voice stops!
- Mass flow reaches a <u>maximum</u> at the throat where $M = 1$

→ <u>Choked flow.</u>

- Can increase area to get more flow.
- Can change the Reservoir to change flow.

$$M = 1 \rightarrow \frac{P_1}{P_R} = \left(\frac{2}{k+1}\right)^{k/k-1} = \underline{\underline{0.53}}$$

$$\frac{T_1}{T_r} = \frac{2}{k+1} = 0.8333 \qquad 25°C \rightarrow -25°C$$

$$\frac{\rho_1}{\rho_R} = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} = 0.634$$

- very common in valves, orifices, vacuum systems
  - bike tire → flow is choked through the tire valve
  - Most Gas Control valves are Choked!
- The Choke point is at the minimum Area in the Nozzle / Valve