ChE 374–Lecture 27–Navier Stokes Equations

- Equations
  - Continuity Equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$
  - Or: $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$ (Cartesian)
  - Or, for constant ρ: $\nabla \cdot \vec{v} = 0$.
  - Momentum: $\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla P - \nabla \cdot \boldsymbol{\tau} + \rho \vec{g}$.
  - Using the product rule and applying the continuity equation:
    $\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P - \nabla \cdot \boldsymbol{\tau} + \rho \vec{g}$.
  - For constant density:
    $\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P + \mu \nabla^2 \vec{v} + \rho \vec{g}$.
  - This can be rewritten with the material derivative as:
    $\rho \frac{D\vec{v}}{Dt} = -\nabla P + \mu \nabla^2 \vec{v} + \rho \vec{g}$.
  - THIS IS THE NAVIER STOKES EQUATION governing fluid flow, which is paired with continuity.
    * Four equations (xyz momentum and continuity) in four unkowns: u,v,w,P).
  - SEE PAGE 450 of your book for this equation in expanded form.

- Boundary Conditions
  - Given velocity at inlet or outlet or far field: v_{in}, v_{out}, v_{∞}.
  - $\frac{\partial v}{\partial x} = 0$ at symmetry points like a pipe centerline.
  - $v = 0$ at walls.

- Initial Conditions
  - For unsteady flows, also specify the initial conditions $\vec{v}(x, y, z)$, $P(x, y, z)$.

- Analytic solutions available only for laminar flows in simple geometries.
- Solve the equations by reducing dimensions and cancelling terms.
- Example: Barometric Equation: no velocity field $\rightarrow \nabla P = -\rho \vec{g}$.
- Example: Couette Flow: Unforced flow between parallel plates, one stationary, the other moving at a fixed speed.
- Example: Flow down an inclined plane.
- Example: Given a 2-D velocity field, compute the pressure field: Book Example 9-13.

Lecture 27 – Navier-Stokes Eqns

Differential Balances

Review

Continuity: $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{v} = 0$; $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$

Const ρ / Unif ρ $\nabla\cdot\vec{v} = 0$; $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Momentum: $\frac{\partial \rho\vec{v}}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}\vec{v}) = \rho\vec{g} - \vec{\nabla}p - \vec{\nabla}\cdot\underline{\underline{\tau}}$

- Specify τ
  - Inviscid $\rightarrow \nabla\cdot\tau = 0$: Euler Eqns
  - Newtonian Fluid: $\nabla\cdot\tau = -\mu\nabla^2 v - \frac{1}{3}\mu\nabla(\nabla\cdot v)$
  - Const / Unif ρ : $\nabla\cdot\tau = -\mu\nabla^2\vec{v}$

Show: BSL App B

Newtonian Fluid, Const ρ $\longrightarrow$ N. S. equation

$$\frac{\partial v}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 v + \vec{g}$$

Boundary Conditions

- Given Velocity at inlet, outlet, far field.
  - V_∞
  - V_{in}
  - V_{out}
- $\frac{\partial v}{\partial x} = 0$, at Symmetry points: like pipe Centerline
  - Symmetry $\rightarrow$ max or min.
- $v = 0$ at walls (no slip)

Initial Conditions

- v field Given at $t = 0$

Example 1 – Barometric Eqn:

- No velocity, just fluid.

$$\frac{\partial \rho \vec{v}}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}\vec{v}) = \rho\vec{g} - \vec{\nabla}P - \vec{\nabla}\cdot\underline{\underline{\tau}}$$

$$\nabla P = -\rho\vec{g}$$

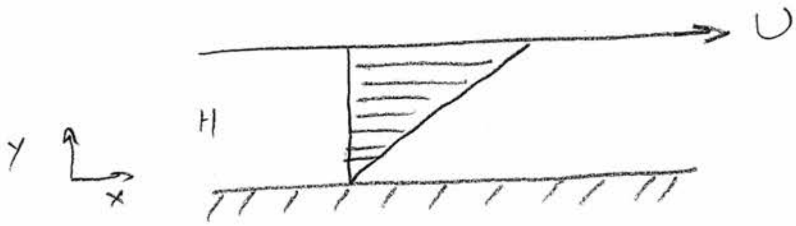
$$\frac{\partial P}{\partial z}\vec{k} = \rho g\vec{k}$$

$$P_2 - P_1 = \rho g(z_2 - z_1)$$

z ↑ —— (1) —— (2)   ↓ h $\Delta P = \rho g h$.

Example 2

Couette Flow



- Const ρ, μ
- BC : $u = 0$ at $y = 0$
  $u = U$ at $y = H$
- SS
- N.S. $$\underbrace{\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}}_{\frac{D\vec{v}}{Dt}} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\vec{v} + \vec{g}$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \vec{V}\cdot\nabla u = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
(SS, M/R, v=0, w=0)

- only x-momentum.

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g_x$$
(SS, open, 1-D, 1-D, 0)

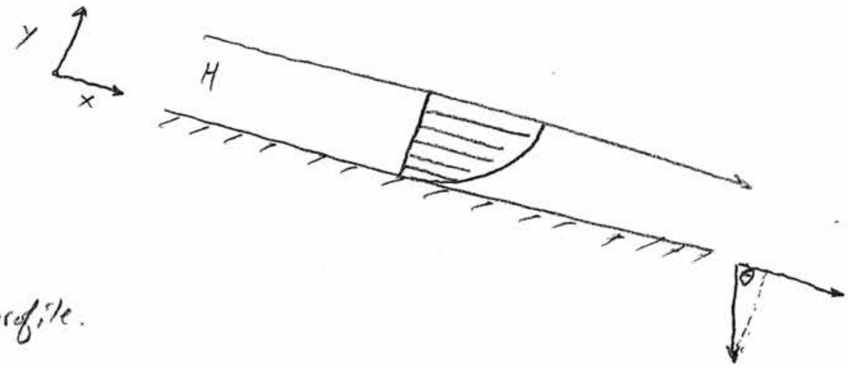
- $\nu\frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow$ no curvature $\longrightarrow$ linear

$u = axtb$; at $y = 0$ $u = 0$ $\longrightarrow b = 0$
at $y = H$ $u = U$ $\longrightarrow a = \frac{U}{H}$

$$\boxed{u = \frac{U}{H}y}$$

Book → Flow w/ $\frac{dP}{dx}$ Given

Example 3



- Flow Down incline.
- Steady
- Laminar
- 1-D (y-Dir)
- $v = w = 0$, → $u(y)$ profile.
- $u(0) = 0$

$$\left.\frac{du}{dy}\right|_{y=H} = 0$$

- N-S. Eq: $$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \vec{V} + \vec{g}$$

- x-mom: $$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g_x$$

SS (steady), open, 1-D, 1-D, $g\cos\theta$

$$\nu \frac{\partial^2 u}{\partial y^2} + g\cos\theta = 0$$

$$\frac{d^2u}{dy^2} = -\frac{g}{\nu}\cos\theta \quad \longrightarrow \quad d^2u = -\frac{g}{\nu}\cos\theta \, dy\, dy$$

$$\xrightarrow{\int} \quad du = \left(-\frac{g}{\nu}\cos\theta(y + c)\right)dy$$

$$\longrightarrow \quad \frac{du}{dy} = -\frac{g}{\nu}\cos\theta\,(y + c)$$

$$y = H \rightarrow \frac{du}{dy} = 0 \longrightarrow c = -H$$

$$du = -\frac{g}{\nu}\cos\theta(y - H)$$

$$\xrightarrow{\int} \quad u = -\frac{g}{\nu}\cos\theta\left(\frac{y^2}{2} - yH + c\right)$$

at $y = 0$ $u = 0$ → $c = 0$

$$\boxed{u = -\frac{g\cos\theta}{\nu}\left(\frac{y^2}{2} - yH\right)}$$

Example 4

Book Example 9-13. → HW P1

- S.S., 2D, incomp. flow.

$u = ax + b$

$v = -ay + cx$

Find $P(x,y)$

X-Mom Eq

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \cancel{\rho g_x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

SS, $a(ax+b)$, 0, 2D; 0, 0, 0

$$\frac{\partial P}{\partial x} = -\rho a^2 x - \rho ab \quad (3)$$

Y-Mom Eq

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \cancel{g\rho} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

SS, $(ax+b)(c)$, $(-a)(-ay+cx)$, 2D; 0

$$\frac{\partial P}{\partial y} = \rho(\cancel{-acx} - bc - a^2 y + \cancel{acx}) \quad (2)$$

$$\frac{\partial P}{\partial y} = -\rho bc - \rho a^2 y$$

integrate → $P(x,y) = -\rho bcy - \frac{\rho a^2 y^2}{2} + g(x)$

$$\frac{\partial P}{\partial x} = g'(x) = -\rho a^2 x - \rho ab$$

$$\int \to g(x) = -\frac{\rho a^2 x^2}{2} - \rho abx + C$$

$$\boxed{P(x,y) = -\rho bcy - \frac{\rho a^2 y^2}{2} - \frac{\rho a^2 x^2}{2} - \rho abx + C}$$

Since eqns only have ∇P in there.

Note The Const C : Generally have This for incomp. flows.