ChE 374–Lecture 26–Differential Balance

- Previously considered Mass, Energy and Momentum balances in INTEGRAL form, using the Reynolds Transport Theorem.
  - Provides overall/bulk properties: Net mass, or net force.
- Differential balance → shrink the Control Volume (C.V.) to a point to get an equation that, when solved, gives the whole flow field.
  - Example was laminar pipe flow: force balances → differential equation.
    * Solved for full pressure and velocity profile.
    * (More involved than usual.)
- Mass Balance: Integral Method
  - Start with integral balance: $\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho\vec{v}\cdot\vec{n}dA = 0$.
  - Gauss Divergence Theorem: $\int_{CS}\vec{v}\cdot\vec{n}dA = \int_{CV}\nabla\cdot\vec{v}dV$.
  - bring the $\frac{d}{dt}$ inside the integral, and apply the GDT to the second term, bringing both resulting integrals over volume into one term:

    $\int_{CV}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v}\right)dV = 0$
  - This holds for any volume so $\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v} = 0$.
- Differential Mass Balance (Continuity Equation): $\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v} = 0$.
- Mass Balance: Differential Method
  - Use accumulation = in - out + (generation=0).
  - Apply to a square with u in and out on left and right at x and x+dx; and v in and out on bottom and top at y and y+dy.
  - Divide through by dx and dy and take the limit as dx and dy → 0.
- Simply Mass Balance:
  - Steady state $\nabla\cdot\rho\vec{v} = 0$ (e.g., a steady flame).
  - Uniform ρ and SS: $\nabla\cdot\vec{v} = 0$.
  - Constant ρ: $\nabla\cdot\vec{v} = 0$.
- Cartesian, Cylindrical, Spherical (different definitions for ∇).
- Continuity equation puts a CONSTRAINT on velocity, its not normally used to Solve for velocity (we use the momentum equation to do that).
- Examples of constraining flow fields using continuity (mass in = mass out).
- Differential Momentum Balance (Integral)
  - Start with Integral Momentum Balance: $\sum\vec{F} = \frac{\partial}{\partial t}\int_{CV}\rho\vec{v}dV + \int_{CS}\rho\vec{v}\vec{v}\cdot\vec{n}dA$.
  - Use $\sum\vec{F} = \int_{CV}]\rho\vec{g}dV - \int_{CS}P\boldsymbol{\delta}\cdot\vec{n}dA - \int_{CS}\boldsymbol{\tau}\cdot\vec{n}dA$.
  - Apply the Gauss divergence theorem, move $\frac{\partial}{\partial t}$ inside the integral and collect the integrals over the control volume. Then apply to any control volume:

    $\frac{\partial\rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = -\nabla P - \nabla\cdot\boldsymbol{\tau} + \rho\vec{g}$.
- Simplify:
  - Inviscid → $\boldsymbol{\tau} = 0$.
  - Constant ρ → $\nabla\cdot\boldsymbol{\tau} = -\mu\nabla^2\vec{v}$.
    * Reduce dimensions.

Lecture 26 - Differential Balances

- Previously: considered Mass, Energy, Mom] Integral Form (RTT)

- Provides overall / Bulk properties
  - Net mass
  - Net Force

- Differential Balance → shrink C.V. to a point to get an equation That, when solved, Gives whole Flow-field

- Integral $\int (\ ) dx$ → Differential $\frac{\partial(\ )}{\partial x}$

- Laminar pipe flow: Saw example of This
  - Derived Differential Eqn.
    - Force Balance → mom Eqn
  - Solved → Full pressure and velocity Profile
  - More involved Than usual.

- Here: Differential Balance for Mass and momentum.

Mass: Method 1

- Start w/ Integral Balance $\frac{d}{dt}\int_{CV} \rho \, d\mathcal{V} + \int_{CS} \rho \vec{V}\cdot\vec{n} \, dA = 0$

- Gauss Div. Theorem

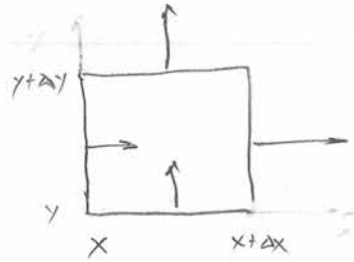
$$\int_A \vec{V}\cdot\vec{n} \, dA = \int_{\mathcal{V}} \nabla\cdot\vec{V} \, d\mathcal{V}$$

$$\rightarrow \frac{d}{dt}\int_{CV} \rho \, d\mathcal{V} + \int \nabla\cdot(\rho\vec{V}) \, d\mathcal{V} = 0$$

$$\rightarrow \int \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{V}\right) d\mathcal{V} = 0$$

$$\rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0}$$

Method 2



Accum = in − out + $\cancel{gen}$

$$\frac{\partial(\rho \Delta x \Delta y)}{\partial t} = (\rho u \Delta y)_x - (\rho u \Delta y)_{x+\Delta x} + (\rho v \Delta x)_y - (\rho u \Delta x)_{y+\Delta y}$$

$\div\ \Delta x, \Delta y$

$$\frac{\partial \rho}{\partial t} = \frac{(\rho u)_x - (\rho u)_{x+\Delta x}}{\Delta x} + \frac{(\rho v)_y - (\rho v)_{y+\Delta y}}{\Delta y}$$

$\lim\ \Delta x, \Delta y \to 0$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho v}{\partial y} \longrightarrow \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\longrightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0}$$

Simplify: S.S. $\longrightarrow \nabla \cdot (\rho \vec{V}) = 0$ (e.g. a steady flame)

Unif ρ, ss $\longrightarrow \nabla \cdot \vec{V} = 0$

Const ρ $\longrightarrow \nabla \cdot \vec{V} = 0$

$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$ is true for any coordinate system.

* Cartesian: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w = 0$

Cylindrical: $\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \rho u_r}{\partial r} + \frac{1}{r}\frac{\partial \rho u_\theta}{\partial \theta} + \frac{\partial \rho u_z}{\partial z} = 0$

Spherical: $\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 \rho u_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \rho u_\phi}{\partial \phi} = 0$

* Note: Continuity Does not <u>Solve</u> for vel, it only Constrains it.

- If Know in → Cont. Says out = in
- If Know one Component Constrains The other.

* Generally, Momentum eqn is used to solve for velocity, Continuity eqn Constrains velocity. → work together.

Example[2]: 2-D ; $u = u(x,y)$
S.S. ; $v = v(x,y)$
Const ρ

Swap

Let $u = ax + by$
$v = cx + dy$

T.P.S. What are Constraints on a, b, c, d ?

$$\nabla \cdot \vec{V} = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rightarrow \frac{\partial(ax+by)}{\partial x} + \frac{\partial(cx+dy)}{\partial y} = 0$$

$$\rightarrow \boxed{a + d = 0}$$

$$\rightarrow \boxed{\text{Pick } a, b, c, d = -a}$$

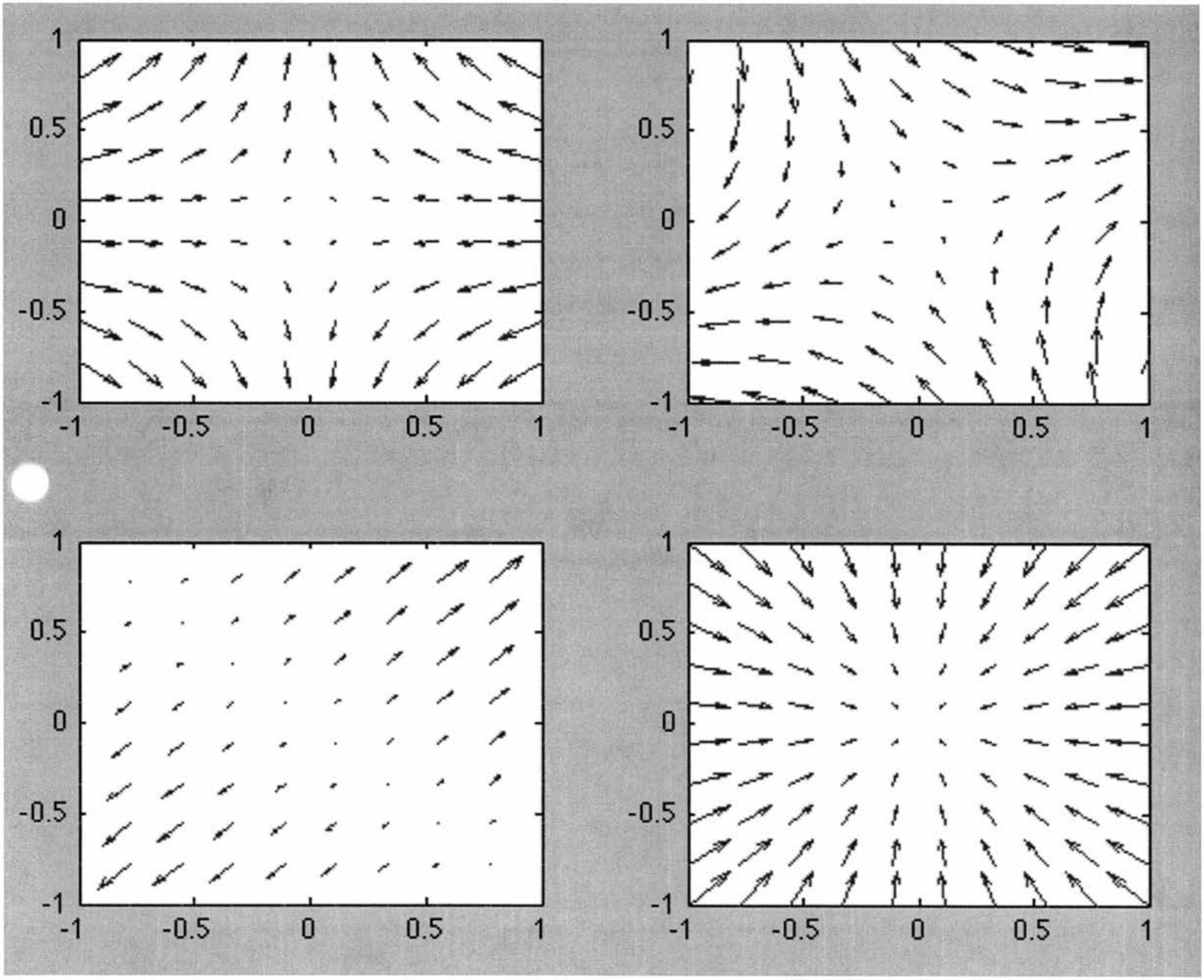
- See 4 plots. vel2.m

Swap Example[1]: Show vel.m → which are possible?

E1


```
clc; clear;
b=1; c=1;

x=linspace(-1,1,10);
y=x;
[X,Y]=meshgrid(x,y);

subplot(2,2,1);
u=-X;
v=Y;
quiver(X,Y,u,v);
axis([-1 1 -1 1]);

subplot(2,2,2);
u=X+Y;
v=X-Y;
quiver(X,Y,u,v);
axis([-1 1 -1 1]);

subplot(2,2,3);
u=X+Y;
v=X+Y;
quiver(X,Y,u,v);
axis([-1 1 -1 1]);

subplot(2,2,4);
u = -X;
v = -Y;
quiver(X,Y,u,v);
axis([-1 1 -1 1]);
```

E1


1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1
0.5
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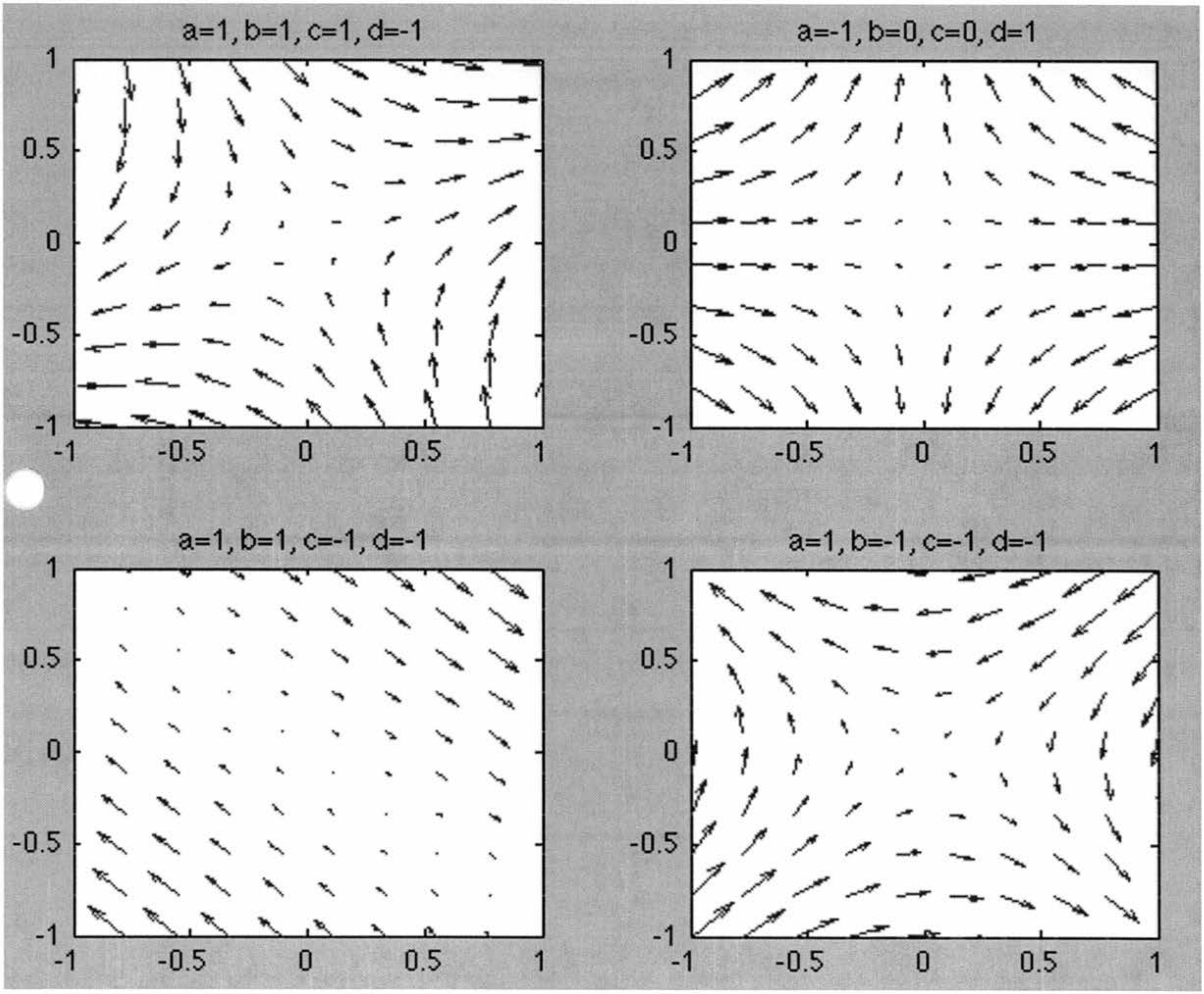
-0.5
-1
-1
-0.5
0
0.5
1
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

E2


```
clc; clear;


x=linspace(-1,1,10);
y=x;
[X,Y]=meshgrid(x,y);

a = 1; b = 1; c = 1; d = -a;
subplot(2,2,1);
u = a*X+b*Y;
v = c*X+d*Y;
quiver(X,Y,u,v);
axis([-1 1 -1 1]);
title('a=1, b=1, c=1, d=-1');

a = -1; b = 0; c = 0; d = -a;
subplot(2,2,2);
u = a*X+b*Y;
v = c*X+d*Y;
quiver(X,Y,u,v);
axis([-1 1 -1 1]);
title('a=-1, b=0, c=0, d=1');

a = 1; b = 1; c = -1; d = -a;
subplot(2,2,3);
u = a*X+b*Y;
v = c*X+d*Y;
quiver(X,Y,u,v);
axis([-1 1 -1 1]);
title('a=1, b=1, c=-1, d=-1');

a = 0; b = -1; c = -1; d = -a;
subplot(2,2,4);
u = a*X+b*Y;
v = c*X+d*Y;
quiver(X,Y,u,v);
axis([-1 1 -1 1]);
title('a=1, b=-1, c=-1, d=-1');
```

E2



Momentum Balance

Integral:

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{cv} \rho \vec{v}\, d\mathcal{V} + \underbrace{\int_{cs} \rho \vec{v}\vec{v}\cdot\vec{n}\, dA}_{\int_{cv} \nabla\cdot(\rho\vec{v}\vec{v})\, d\mathcal{V}}$$

G.DT:

$$\sum \vec{F} = \int_{cv} \rho\vec{g}\, d\mathcal{V} - \underbrace{\int_{cs} P \underline{\underline{\delta}}\cdot\vec{n}\, dA}_{-\int_{cv} \nabla P\, d\mathcal{V}} - \int_{cs} \underline{\underline{\tau}}\cdot\vec{n}\, dA$$

$$- \int_{cv} \nabla\cdot\underline{\underline{\tau}}\, dV$$

Collect

$$\rightarrow \boxed{\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = -\nabla P - \nabla\cdot\underline{\underline{\tau}} + \rho\vec{g}}$$

Combine w/ Mass Balance.

$$\boxed{\rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla P - \nabla\cdot\underline{\underline{\tau}} + \rho\vec{g}}$$

$\nabla\cdot\underline{\underline{\tau}}$ is the complex term $\Rightarrow$ simplify.

1. Inviscid: $\rightarrow$ no τ term (often works away from walls)

2. Const ρ: $\nabla\cdot\underline{\underline{\tau}} = -\mu\nabla^2\vec{v}$; $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

3. Reduce Dimensions:

   Pipe flow $\rightarrow$ $u(y)$, $v = w = 0$

Mass $$\nabla \cdot \vec{v} = 0$$

Momentum $$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P + \rho \vec{g} - \nabla \cdot \tau$$

Mass $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

X-Mom $$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Y-Mom $$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Z-Mom $$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$