Lecture 15 - Dimensional Analysis and Scaling

Overview :
- Dimensionless Groups
- Example
- Scaling

Nondimensionalize an equation

$$\frac{d\rho}{dt} + \frac{\partial \rho v}{\partial x} = 0 \qquad (=) \frac{kg}{m^3 \cdot s} + \frac{kg}{m^3}\frac{m}{s}\cdot\frac{1}{m}$$

(1) each Term must have the Same units → Dimensional Homogeneity.

(2) Non-Dimensionalize By ÷ eqn By Quantity with The Same units of Terms.

→ $\div \rho/t$ (or Some ρ/t or equivalent)

Another Π - Group Example

Book - Falling Sphere : $\frac{d^2 z}{dt^2} = -g$ w/ $z = z_0$ at $t = 0$

Params : z, t, g, z_0, V_0
$m \quad s \quad \frac{m}{s^2} \quad m \quad \frac{m}{s}$

$n_{vars} = 5$
$j_{Dim} = 2$ ⟶ $k_\pi = 5 - 2 = 3$ Π's

Find 3 Π's : Inspection → $\boxed{\frac{z}{z_0} = \Pi_1}$

→ $\boxed{t V_0 / z_0 = \Pi_2}$

Now need include g

$$g (=) \frac{m}{s^2} \div V_0 = \frac{g}{V_0} (=) \frac{1}{s} \times \frac{z_0}{V_0} = \boxed{\frac{g z_0}{V_0^2} = \Pi_3}$$

$$\Pi_3^{-1/2} = \frac{V_0}{\sqrt{g z_0}} = Fr$$

$\Pi_1 = \frac{z}{z_0}$ ratio of lengths

$\Pi_2 = t \frac{V_0}{z_0}$ ratio of times

$\Pi_3 = \frac{g z_0}{V_0^2}$ ratio of inertial and gravity forces.

Note: can form a time out of g $\longrightarrow t \frac{g}{V_0} = \Pi_2$

can form a length out of g $\longrightarrow \frac{z g}{V_0^2} = \Pi_1$

leftover $\longrightarrow \frac{z}{z_0} = \Pi_3$

$\rightarrow$ Π's not unique

Scaling: See Slides.

- Recall: Nature does not "know" about our units.
  - kilogram, pounds
  - meters, miles
  - seconds, hours
- Nature only knows about fundamental dimensions, and relative sizes with those dimensions.
  - When we assign values to those dimensions or combinations of those dimensions, we do so relative to our own contrived scales.
  - **Most problems have more natural scales than our units (1 m).**
  - What is the most natural length scale for flow in a room?
    - Then measure length in "room" units → x/L_{room}.

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How long to cook a roast?

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- How long to cook?
  - Trial and error?
  - Approximate the geometry and properties and solve a heat transfer problem?
  - Solve a complex numerical solution?
- Cookbook Instructions call for **~20 min per lb.**
- Does this make sense? Why or why not?

> www.ochef.com
>
> "If you roast at 325°F (160°C), subtract 2 minutes or so per pound. If the roast is refrigerated just before going into the oven, add 2 or 3 minutes per pound. ...
>
> "...You thought this was going to be a simple answer, didn't you? "

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What does the heat equation say?

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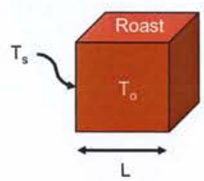
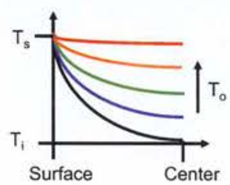
$$\frac{\partial T}{\partial t} = -\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

$$\frac{\partial T}{\partial t} = -\alpha\left(\frac{\partial^2 T}{\partial x_i^2}\right)$$

$$\alpha = \frac{k}{\rho c_p}$$

(And 1 IC, 2 BCs)





- Unsteady heat equation
  - Assuming constant thermal conductivity
  - 3 dimensional, Cartesian coordinates
- Dimensional homogeneity: each term has the same **dimensions** and the same **units**.
  - k→ J/s*m*K; ρ → kg/m^3; c_p→J/kg*K; α→m^2/s
- We don't want to solve this equation, just examine it.
- Nondimensionalize, then scale the equation.

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Nondimensionalize

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$$t^* = \frac{t}{\tau} \rightarrow t = \tau t^*$$
$$T^* = \frac{T}{T_{ref}} \rightarrow T = T_{ref} T^*$$
$$x^* = \frac{x}{L} \rightarrow x = L x^*$$

$$\longrightarrow \frac{\partial T}{\partial t} = -\alpha \left(\frac{\partial^2 T}{\partial x_i^2}\right) \longrightarrow \left[\frac{T_{ref}}{\tau}\right]\left(\frac{\partial T^*}{\partial t^*}\right) = -\left[\frac{\alpha T_{ref}}{L^2}\right]\left(\frac{\partial T^*}{\partial x^{*2}}\right)$$

- Select reference quantities: T_{ref}, L, τ
- Make a direct substitution.
- Simplify: divide through by T_{ref}/τ $\qquad \frac{\partial T^*}{\partial t^*} = -\frac{\alpha \tau}{L^2}\left(\frac{\partial T^*}{\partial x^{*2}}\right)$
- Now all terms are nondimensional.
- The group $\alpha\tau/L^2$ is called a dimensionless group and is the Fourier number: a ratio of the physical time to the characteristic diffusion time.

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Scaling

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- Now scale the equation:
  - Cengel and Boles call this "normalization"
- If the terms in (…) are O(1), then since the LHS is O(1), the term in […] must be O(1) also. Then what to choose for τ, L?
  - L is just the domain size.
  - Then $\tau = L^2/\alpha$
- These are the characteristic scales of the problem.
- Rather than measure length in units of "meters", nature prefers units of "roast size"

$$\left[\frac{T_{ref}}{\tau}\right]\left(\frac{\partial T^*}{\partial t^*}\right) = -\left[\frac{\alpha T_{ref}}{L^2}\right]\left(\frac{\partial T^*}{\partial x^{*2}}\right)$$

$$\left(\frac{\partial T^*}{\partial t^*}\right) = -\left[\frac{\alpha \tau}{L^2}\right]\left(\frac{\partial T^*}{\partial x^{*2}}\right)$$

"… in the process of scaling, one attempts to select intrinsic reference quantities so that each term in the dimensional equations transformes into the product of a constant factor which closely estimates the term's order of magnitude and a dimensionless factor of unit order of magnitude"

-- Lin and Segal 1988

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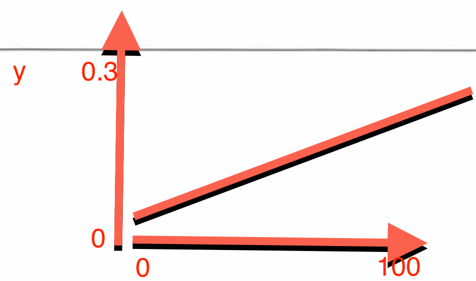


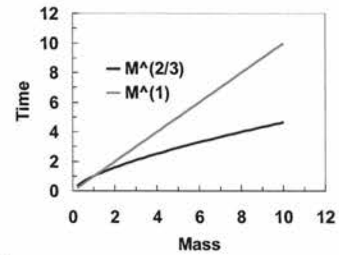
dy/dx -->
dy*/dx*:
Y* = y/yref
x* = x/xref

dy* / dx* = O(1)

xref=100

- Now what does this say about cooking a roast?
- The timescale scales as L^2
- But mass scales as L^3

$$\tau \propto L^2, \quad m \propto L^3 \rightarrow L \propto m^{1/3} \quad \Rightarrow \quad \tau \propto m^{2/3}$$



- So cooking time scales with $m^{2/3}$, not with m
- We have not had to solve anything, but we know something valuable about the problem.
  - If you know the time for one roast, you can extrapolate to another.
  - Relationships among parameters!

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- Nondimensionalization also reduces the number of parameters by showing that they are not independent, but come in characteristic groups.
  - Conduction problem: vary the Fourier number alone, not alpha and L separately.
  - Book gives simple equation of motion, where initial position, velocity and gravity parameters are collapsed into one paramter, the Foude number.
  - The Fourier number was the ratio of timescales, the Froude number is the ratio of forces.
  - Many others: Friction factor, Drag coefficient, Knudsen number, Mach number, REYNOLDS number.

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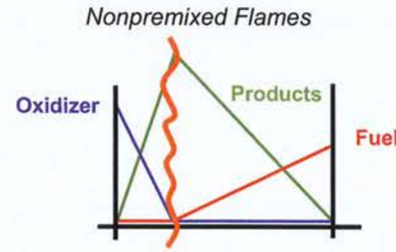
Scaling Example – One Step Combustion



SPECIES

H2	CH2	CH3OH	AR	n-C4H3	C6H3	C5H4OH
H	CH2Z	C2H	N2	i-C4H3	l-C6H4	C5H4O
O	CH3	C2H2	CH2CHO	C4H4	c-C6H4	C3H8
O2	CH4	C2H3	C3H2	n-C4H5	A1	nC3H7
OH	CO	C2H4	C3H3	i-C4H5	A1-	iC3H7
H2O	CO2	C2H5	pC3H4	C4H6	C6H5O	C3H6
HO2	HCO	C2H6	aC3H4	C4H612	C6H5OH	aC3H5
H2O2	CH2O	HCCO	cC3H4	C4H81	C5H6	CH3CCH2
C	CH2OH	CH2CO	C4H2	C4H7	C5H5	CH3CHO
CH	CH3O	HCCOH	H2C4O	C6H2	C5H5O	C2H3CHO

(k = A T**b exp(-E/RT))

REACTIONS	A	b	E
1. 2O+M <=> O2+M	1.20E+17	-1.0	0.0
2. O+H+M <=> OH+M	5.00E+17	-1.0	0.0
3. O+H2 <=> H+OH	3.87E+04	2.7	6260.0
4. O+HO2 <=> OH+O2	2.00E+13	0.0	0.0
5. O+H2O2 <=> OH+HO2	9.63E+06	2.0	4000.0
6. O+CH <=> H+CO	5.70E+13	0.0	0.0
7. O+CH2 <=> H+HCO	8.00E+13	0.0	0.0
8. O+CH2Z <=> H2+CO	1.50E+13	0.0	0.0
9. O+CH2Z <=> H+HCO	1.50E+13	0.0	0.0
.....			
.....			
.....			
457. C4H7+H <=> C4H6+H2	1.80E+12	0.0	0.0
458. C4H7+O2 <=> C4H6+HO2	1.00E+11	0.0	0.0
459. C4H7+HO2 <=> CH2O+OH+aC3H5	2.40E+13	0.0	0.0
460. C4H7+HCO <=> C4H81+CO	6.00E+13	0.0	0.0
461. C4H7+CH3 <=> C4H6+CH4	1.10E+13	0.0	0.0
462. C2H3+C2H4 <=> C4H7	7.93E+38	-8.5	14220.0
463. C2H3+C2H5(+M) <=> C4H81(+M)	1.50E+13	0.0	0.0

- Large-scale jet flame simulations: 1-2 weeks on 10,000 processors
- Ethylene combustion with complex chemistry: 19 species and 167 chemical reactions
- Speed setup with a simpler 1-step chemical reaction.
- Reaction characteristics?
- Beauty and betrayal in this equation

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Apply Characteristic Timescale

$$C_2H_4 + 3O_2 \rightarrow 2CO_2 + 2H_2O$$

$$\frac{d[C_2H_4]}{dt} = -k(T)[C_2H_4]^{0.1}[O_2]^{1.65}$$

$$C_2H_4 + 3O_2 \rightarrow 2CO_2 + 2H_2O$$

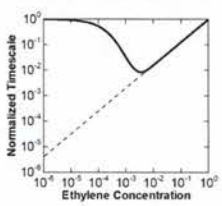
$$[C_2H_4]^{0.1} \rightarrow [C_2H_4]^{0.1+0.9\exp(-800Y_{C2H4})}$$

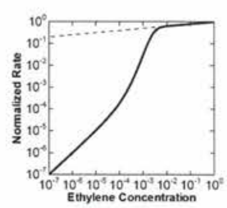
- ***Problem:*** Very Stiff Explicit ODE integration
  - Very small timestep sizes required.
  - Takes too many steps.
  - Why?
- Apply Characteristic Timescale

$$\tau = \frac{\Delta[C_2H_4]_{max}}{\left|\frac{d[C_2H_4]}{dt}\right|_{max}} \qquad \tau = \frac{[C_2H_4]}{[C_2H_4]^{0.1}} = [C_2H_4]^{0.9}$$

- As $[C_2H_4]$ becomes small, τ becomes small.
- The small reaction order is the culprit.
- Fix: As [C2H4] becomes small, increase reaction order.

$$\tau_{new} = [C_2H_4]^{0.9[1-\exp(-800Y_{C2H4})]}$$





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