Chemical Engineering 374

Fluid Mechanics

Dimensional Analysis

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Outline

- Introduction and Motivation
  - Theory
    - Limitations
  - Experiments
    - Cost, practicality, efficiency
    - Qualitative understanding
- Dimensionless groups
  - Eliminate the units
  - Methods
    - Governing Equations
    - Force Ratios
    - **Π Method**
  - Examples
- Similarity / Scale models

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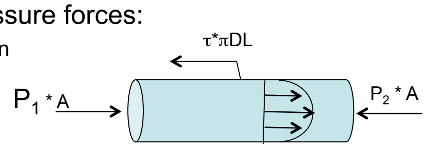
Pipe Flow

- Consider flow in a pipe
  - (We'll do lots of this!)
- So far, mostly frictionless flow
  - (Friction next week)
  - But we've discussed friction on a plate for Newton's law of viscosity: $\tau = \mu * du/dr$.
  - Friction force is balanced by pressure forces:
    - Pressure drop Δp balanced by friction



- Most flows are turbulent (consider averages)
- **Question/Goal:** How to characterize pipe flow?

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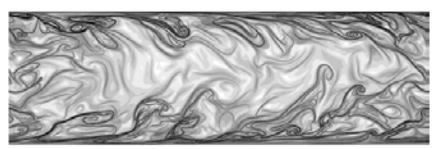
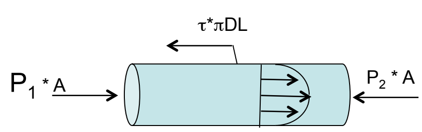
The Challenge

- Turbulent flow is complex
  - Random, Chaotic
- How to measure and relate the important properties...
  - List them:





- V, ΔP, L, D, μ, ρ

Or

- V, $\Delta P/L$, D, μ, ρ

Solutions

- Solve Governing Equations?

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} &= 0 \\
\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} + \frac{\partial (\rho uw)}{\partial z} &= -\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\frac{\partial \rho v}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho v^2)}{\partial y} + \frac{\partial (\rho vw)}{\partial z} &= -\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
\frac{\partial \rho w}{\partial t} + \frac{\partial (\rho uw)}{\partial x} + \frac{\partial (\rho vw)}{\partial y} + \frac{\partial (\rho w^2)}{\partial z} &= -\frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \\
\frac{\partial \rho E}{\partial t} + \frac{\partial (\rho uE)}{\partial x} + \frac{\partial (\rho vE)}{\partial y} + \frac{\partial (\rho wE)}{\partial z} &= -\frac{\partial pu}{\partial x} - \frac{\partial pv}{\partial y} - \frac{\partial pw}{\partial z} + S
\end{aligned}
$$

  - (Yes, but its expensive!)
- Experiments?
  - Design Parameters?
  - Span the Space.
  - How many experiments to do?
    - Measure $\Delta P/L$, for a range of D, v, μ, ρ?
    - If 10 points in each of D, v, μ, ρ then have 10,000 experiments to do
  - Lets be smart! (be efficient)
- Dimensional analysis

Dimensional Analysis

- Dimensional analysis is a powerful tool
  - Insight into equations
  - Relationships among parameters
  - Reduce the number of parameters in a system
  - Obtain scaling laws
- Concept: Nature does not "know" about our units.
  - kilogram, pounds
  - meters, miles
  - seconds, hours
- Nature only knows about fundamental dimensions, and relative sizes with those dimensions.
  - When we assign values to those dimensions or combinations of those dimensions, we do so relative to our own contrived scales.

Find Dimensionless Groups

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- Not all parameters are independent
- Pipe flow:
  - A flow with v=4, D=1 is the SAME, as a flow with v=2, D=2
    - (all else equal).
  - A flow with μ=20, ρ=1.2 is the SAME as a flow with μ=40, ρ=2.4
- Find the independent parameters, the dimensionless groups.
- 3 Methods
  - Governing Equations
  - Force Ratios
  - Π method

Governing Equations

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- Find relevant dimensionless groups from a governing equation.
- Bernoulli equation: $\frac{P}{\rho} + \frac{v^2}{2} + gz = C$
  - Divide by one of the terms like $v^2/2$

$$\underbrace{\frac{P}{\rho v^2/2}}_{C_p} + 1 + \underbrace{\frac{gz}{v^2/2}}_{2/Fr^2} = \frac{C}{v^2/2}$$

Force Ratios

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- Take important forces in a problem and form ratios of them
- B.E. → pressure, gravity, momentum (velocity) forces.
  - Pressure → P*A
  - Gravity → mg = ρAzg
  - Momentum → $\rho A v^2/2$
    - (recall stagnation flow converts pressure force to velocity)
  - Form ratios → same as for method of governing equations

Π Method

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- General and systematic approach
- Simple to do, but lots of conditions and rules to be general
  - WARNING, your book is very confusing here.
- Method for pressure drop in a pipe.
  1) List the parameters and variables, along with their symbols:
     - ΔP/L $kg/m^3 s^2$
     - D   m
     - ρ kg/m^3
     - μ   kg/m*s
     - v   m/s

     → n_{var}=5 variables
  2) Count the number of dimensions → j_{dim} = 3   (kg, m, s)
  3) # of Π's → $k_\Pi = n_{var} - j_{dim}$ = 5-3 = 2
     - That is, I have 5 vars, but how many are independent? Nature doesn't regard our units, so, the problem has to be non-dimensional, so I have 5 variables, but I have to get rid of 3 dimensions (units), so I have 3 constraints → 5-3 = 2 independent variables → 2 Π's.
  4) Now find the Π's
     - Need to include all variables among the Π's
     - The Π's need to be independent
     - Usually can find the Π's by trial (and error)
     - Be smart: if you want to find a relationship for ΔP, then don't put ΔP in all the Π's.

Get the Π's

- Vars:
  - ΔP/L kg/m^3*s^2
  - D   m
  - ρ kg/m^3
  - μ   kg/m*s
  - v   m/s
- Π_1:
  - Start with ΔP/L and get rid of its units using the other variables:

$$\frac{\Delta P}{L}\,(=)\,kg/m^2s^2 \;|\; \div\rho \rightarrow \frac{\Delta P}{\rho L}\,(=)\,m/s^2 \;|\; \div v^2 \rightarrow \frac{\Delta P}{\rho v^2 L}\,(=)\,1/m \;|\; \times D \rightarrow \frac{D\Delta P}{\rho v^2 L}$$

- Now try Π_2:

$$\mu\,(=)\,kg/ms \;|\; \div\rho \rightarrow \frac{\mu}{\rho}\,(=)\,m^2/s \;|\; \div v \rightarrow \frac{\mu}{\rho v}\,(=)\,m \;|\; \div D \rightarrow \frac{\mu}{\rho v D}$$

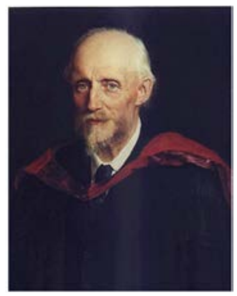
- The Π's are nondimensional so they are not uniq:
  - $\Pi_2 \leftarrow \Pi_1\Pi_2$ is okay (that is, replace Π_2 with $\Pi_1\Pi_2$)
  - Π to any power is okay → our $1/\Pi_2$ above IS VERY SPECIAL: **Re = ρDv/μ**
  - Π_2 is also special (but we'll talk about it later).

More General

- We found Π by inspection, (pretty easy).
- "Repeating variables" approach (book gives lots of rules):
  - n_{var} = 5, j_{dim}=3, k_{Π}=2
  - Pick j_{dim} repeating vars that will show up in all Πs
    - D, ρ, v
  - Form the two Πs using the two leftover vars (one var in each Π)
    - ΔP/L and μ
  - Π_1: $$\frac{\Delta P}{L}\cdot D^a\rho^b v^c \;(=)\; kgm^{-2}s^{-2}\cdot m^a kg^b m^{-3b}\cdot m^c s^{-c}$$ $$(=)\; kg^{1+b}\cdot m^{a+c-3b-2}\cdot s^{-2-c}$$
    - Now, select a, b, c so units cancel (powers=0) → $kg^0 = 1$
    - kg: 1+b = 0
    - m: a+c-3b-2 = 0
    - s: -2-c = 0

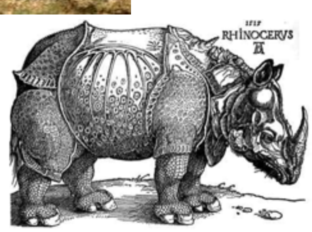
→ 3 eqn in 3 unknowns → a=1, b=-1, c=-2
→ $\Pi_1 = D\Delta P/\rho L v^2$ as before.

- Dimensionless groups reprent the ratio of two physical phenomena.
- Reynolds number
  - Most important in fluid mechanics
  - Re = $\rho Lv/\mu$.
  - Ratio of inertial and viscous forces
    - (or timescales or lengthscales).
- Osborne Reynolds: 1842-1912
  - British engineer.
  - Many advances in fluid mechanics
  - Pipe flow: laminar/turbulent transition.



Applications—Similarity

- Why do animals have the shape and size they do?
- Are giants are practical?



Similarity

- As chemical engineers, we design processes and plants.
  - Use governing equations.
  - These equations are often inadequate
    - Can't always solve.
    - Don't always know the equations.
    - Reality is often too complex.
  - Do experiments.
    - Cost is high → small scale, then scale up.
    - Make sure consistent at the two scales.
    - "Have your failures on a small scale, in private; have your successes on a large scale, in public!"
  - 3 similarity requirements
    - Geometric (shape)
    - Kinematic (velocities)
    - Dynamic (Forces)
  - Find the dimensionless groups, and make sure they are the same for the model and the scale versions.
    - Dimensionless groups don't need the full model, only the important parameters.