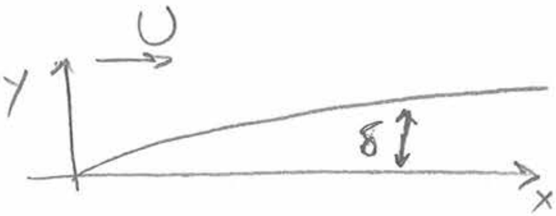
Boundary - Layer Eqns - Flat plate.



$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

BC $\quad y=0 \rightarrow u=v=0$

$\quad\quad y=\infty \rightarrow u=U$

Similarity

- No preferred length
- If scale u by U, y by δ, then the profile looks the same at all x locations.

Then $\dfrac{u}{U} = f\left(\dfrac{y}{\delta}\right) = f(\eta)$

$$\eta = \frac{y}{\delta} = y\sqrt{\frac{U}{x\nu}}$$

$$\frac{\delta}{L} = \sqrt{\frac{1}{Re}} \rightarrow \delta = \sqrt{\frac{L\nu}{U}} \quad ; \quad L \text{ is } x$$

$$* \ (1) \quad u = Uf\left(y\sqrt{\frac{U}{x\nu}}\right) = Uf(\eta)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = U\left[\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}dx + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}dy\right]$$

$$du = U\frac{\partial f}{\partial \eta}\left[-\frac{1}{2}\frac{y}{x}\sqrt{\frac{U}{x\nu}}dx + \sqrt{\frac{U}{x\nu}}dy\right]$$

$$* \ (2) \quad \frac{\partial u}{\partial x} = -U\frac{\partial f}{\partial \eta}\cdot\frac{1}{2}\frac{y}{x}\sqrt{\frac{U}{x\nu}}$$

$$* \ (3) \quad \frac{\partial u}{\partial y} = U\frac{\partial f}{\partial \eta}\sqrt{\frac{U}{x\nu}}$$

$$* \ (4) \quad \frac{\partial^2 u}{\partial y^2} = \frac{U^2}{x\nu}\frac{d^2 f}{d\eta^2}$$

Sub these into the Mom Eqn.

Use Continuity to get v

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{Uy}{2x}\sqrt{\frac{U}{x\nu}}\,\frac{df}{d\eta}$$

$$\partial v = \frac{Uy}{2x}\sqrt{\frac{U}{x\nu}}\,\frac{df}{d\eta}\,\partial y \qquad ; \qquad y = \eta\sqrt{\frac{x\nu}{U}}$$

$$\partial y = \sqrt{\frac{x\nu}{U}}\,\partial\eta$$

$$\partial v = \frac{U}{2x}\,\frac{df}{\cancel{d\eta}}\,\cancel{\partial\eta}\cdot\eta\sqrt{\frac{x\nu}{U}}$$

$$\partial v = \frac{U\eta}{2x}\sqrt{\frac{x\nu}{U}}\,df$$

$$\int \longrightarrow \quad v = \frac{U}{2}\sqrt{\frac{\nu}{xU}}\int \eta\, df$$

$$\int u\,dv = uv - \int v\,du$$

$$v = \frac{U}{2}\sqrt{\frac{\nu}{xU}}\left(\eta f - \int f\,d\eta\right)$$

let $g = \int f\,d\eta \longrightarrow f = g' \longrightarrow f' = g'' \longrightarrow f'' = g'''$

*(4) $$v = \frac{1}{2}\sqrt{\frac{U\nu}{x}}\left(\eta g' - g\right)$$

Sub into Mom Eq

$$-Ug'\left(\frac{\overset{\eta}{\cancel{y}}}{2x}U\sqrt{\frac{U}{x\nu}}\,g''\right) + \frac{1}{2}\sqrt{\frac{U\nu}{x}}\left(\eta g' - g\right)U g''\sqrt{\frac{U}{x\nu}} = \nu\frac{U^2}{x\nu}g'''$$

$$y = \eta\sqrt{\frac{x\nu}{U}}$$

$$-\frac{\cancel{U^2}}{2\cancel{x}}\eta g'' + \frac{1}{2}\cdot\frac{\cancel{U^2}}{\cancel{x}}\left(\eta g' - g\right)g'' = \frac{\cancel{U^2}}{\cancel{x}}g'''$$

$$-\cancel{\eta g' g''} + \cancel{\eta g' g''} - gg'' = 2g'''$$

*(6) $\boxed{gg'' + 2g''' = 0}$ $\quad \boxed{g = g(\eta)\ ;}$

(1) BC: $y=0 \rightarrow u=0$
$\rightarrow \eta=0,\ f=g'=$

(5) BC: $y=0 \rightarrow v=0 \rightarrow \eta=$
$g=0$

(1) BC: $y=\infty \rightarrow u=U$
$\eta=\infty,\ f=g'=1$

Now Solve

$$\begin{cases} g g'' + 2 g''' = 0 \end{cases}$$

$\eta = 0 \quad g' = 0$

$\eta = 0 \quad g = 0$

$\eta = \infty \quad g' = 1$

let $h = g' \rightarrow g'' = h', \quad g''' = h''$

$$\begin{cases} g h' + 2 h'' = 0 \\ g' = h \end{cases}$$

let $k = h' \rightarrow h'' = k'$

$$\begin{cases} g h' + 2 k' = 0 \\ g' = h \\ h' = k \end{cases}$$

or
$$\left.\begin{matrix} g k + 2 k' = 0 \\ g' = h \\ h' = k \end{matrix}\right\} \rightarrow \begin{bmatrix} k' = -\frac{1}{2} g k \\ h' = k \\ g' = h \end{bmatrix}$$

$\eta = 0, \quad h = 0$

$\qquad\quad g = 0$

$\eta = \infty \quad h = 1$

Guess $k(\eta = 0)$

Solve $\rightarrow h(\eta = \infty)$

repeat to converge $h(\eta = \infty) = 1$

$$\boxed{\frac{u}{U} = h(\eta)}$$

Use a shooting method w/ Newton's method.

Converge $F(k_0) = h_\infty - 1 = 0$

(Plug in k_0, Get out $h(\eta = \infty)$; Solve root $F(\cancel{k_\infty}) = 0$)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%  FILE bbl.m %%%%%%%%%%%%%%%%%%%%%%%%%%%%
% D. O. Lignell
% 11/4/09
% Solution to the Blasius B.L.
% u/U = f(eta);
% eta = y/delta = y*sqrt(U/x/nu)
%
% Governing equation: g * g'' + 2 * g''' = 0;
% Boundary conditions:
%      eta=0, g'=0;          ( y=0, u=0 )
%      eta=0, g=0;           ( y=0, v=0 )
%      eta=infty, g'=1;      ( y=infty, u=U)
% Here, f = g', f' = g'', f'' = g'''
%
% Rewrite this single ode as 3 first order ODE's:
% g' = h
% h' = k
% k' = -0.5*g*k
% Boundary conditions:
% eta = 0     --> h=g=0;
% eta = infty --> h=1;

% Solve with a shooting method
% (guess initial k, solve as IVP, find h at eta=infty, guess new initial k, repeat)
% (use Newton's method for the convergence).
% (or, k(eta=0) = 0.33206 is the solution)

clc;
clear;

global etaRange;

etaRange = [0 10];              % 0-10 is like 0-infinity (by trial)

%---------- Converge the initial condition for k

kg = 0.0                        % initial guess for k(eta=0)

for iter=0:5;                   % just do 5 Newton iterations
  kg = kg - F(kg)/(dFdk(kg))
end

%---------- Now we know kg = k(eta=0), so solve IVP

ghk0 = [0 0 kg];                             % IC for g, h, k
[eta ghk] = ode45(@rhs, etaRange, ghk0);     % Solve the system

if( abs(ghk(end,2)-1) > 0.000001 )
    error('Not converged, use more iterations');
end

%---------- recover u/U = f = g' = h

u_U = ghk(:,2);                 % u_U is u/U

%---------- Plot results

plot(u_U, eta);
ylabel('eta');
xlabel('u/U');
title('Flat Plate Boundary Layer Solution');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% FILE F.m   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function y = F(kg)

global etaRange;

    ghk0 = [0 0 kg];
    [eta ghk] = ode45(@rhs, etaRange, ghk0);   % Solve the system

    y = ghk(end,2) - 1;       % h(eta=infty)-1 = F(kg) = 0 when get kg
```

```
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% FILE rhs.m    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function r = rhs(eta, y)

    g = y(1);
    h = y(2);
    k = y(3);

    r = zeros(3,1);
    r(1) = h;
    r(2) = k;
    r(3) = -1/2*g*k;

end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% FILE dFdk.m    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Simple numerical derivative;

function y = dFdk(kg)

global etaRange;

    dkg = 0.000001;
    y = (F(kg+dkg) - F(kg)) / dkg;

end
```

MATLAB 7.6.0 (R2008a)

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s/Lecture_28_BoundaryLayers

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```
>> bbl

kg =

     0

kg =

   0.100002083321851

kg =

   0.283627555985008

kg =

   0.330761625044068

k

   0.332042293290865

kg =

   0.332043118116658

kg =

   0.332043118116586

>>
```

Figure 1

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