Solving Basic Rate Equations

- Rate Equations = Ordinary Differential Equations (ODEs)
  - You will take a class on solving these analytically
  - Here will learn basic techniques for solving them numerically
- Basic form

$$\frac{dy}{dt} = f(y,t)$$

- dy/dt = y'
- We are **given** the "slope" y' = f(y,t).
- We want to **find** the function y(t).

This is NOT a solver problem.
y(t) is an unknown function, not a value.
We want y(t) at all points t.
We are given y'(y,t), the slope.

Initial Condition

- Conceptually, we can solve this by integrating.
- There is a constant of integration.
- We evaluate the constant by specifying an initial condition: $y(0) = y_0$
- Recall, adding any constant to y doesn't change the rate equation

$$\frac{d(y+c)}{dt} = \frac{dy}{dt} + \underbrace{\frac{dc}{dt}}_{0} = \frac{dy}{dt} = f(y,t)$$

2 Things: rate function f(y,t), and an initial condition

$$\frac{dy}{dt} = f(y,t)$$
$$y(0) = y_0$$

Example

- **F = ma**
- (Rate of change of momentum) = (sum of external forces)

$$\frac{d(mv)}{dt} = mg$$

Gravitational force

- For constant mass, we have

$$\frac{d(v)}{dt} = g$$

- Initial condition

$$v(0) = v_0$$

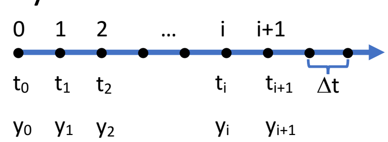
Solve for v(t)

$$v(t) = gt + v_0$$

This is analytic, we'll do it numerically

Explicit Euler Method

- Don't solve analytically, solve numerically
- Solve only at discrete points t_i
- Approximate the slope



$$\frac{dy}{dt} = f(y,t)$$

$$\frac{\Delta y}{\Delta t} = f(y,t)$$

$$\frac{y_{i+1} - y_i}{\Delta t} = f(y_i, t_i)$$

$$y_{i+1} = y_i + \Delta t f(y_i, t_i)$$

Start i=0, y=y_0

$$t_i = i\Delta t$$

$y_1 = y_0 + \Delta t f(y_0, t_0)$ Step from i=0, to i=1

$y_2 = y_1 + \Delta t f(y_1, t_1)$ Step from i=1, to i=2

$\ldots$

$y_{i+1} = y_i + \Delta t f(y_i, t_i)$ Step from i, to i+1, etc.

Example

- Nuclear decay.
- The rate of loss is proportional to the amount.
- Here, f(c,t) = -kc
  - We include the “t” dependence in writing the function, but in many cases “t” does not explicitly appear in the formula.
- Explicit Euler equation $y_{i+1} = y_i + \Delta t f(y_i, t_i)$

$$\frac{dc}{dt} = -kc$$

$$c(0) = c_0$$

$$c_{i+1} = c_i + \Delta t \cdot (-kc_i)$$

- Solve in Excel…

Excel…

2 Rate Equations?

$$y_{i+1} = y_i + \Delta t f(y_i, z_i, t_i)$$
$$z_{i+1} = z_i + \Delta t g(y_i, z_i, t_i)$$

*Both y, and z are written in terms of the **previous** point*

$$\frac{dy}{dt} = f(y, z, t)$$
$$\frac{dz}{dt} = g(y, z, t)$$

Example: falling raindrop with air resistance

$$\frac{dv}{dt} = g - cv^2 \qquad \frac{dx}{dt} = v$$

$\Rightarrow$

$$v_{i+1} = v_i + \Delta t(g - cv_i^2)$$
$$x_{i+1} = x_i + \Delta t(v_i)$$

$$y(0) = y_0$$
$$z(0) = z_0$$