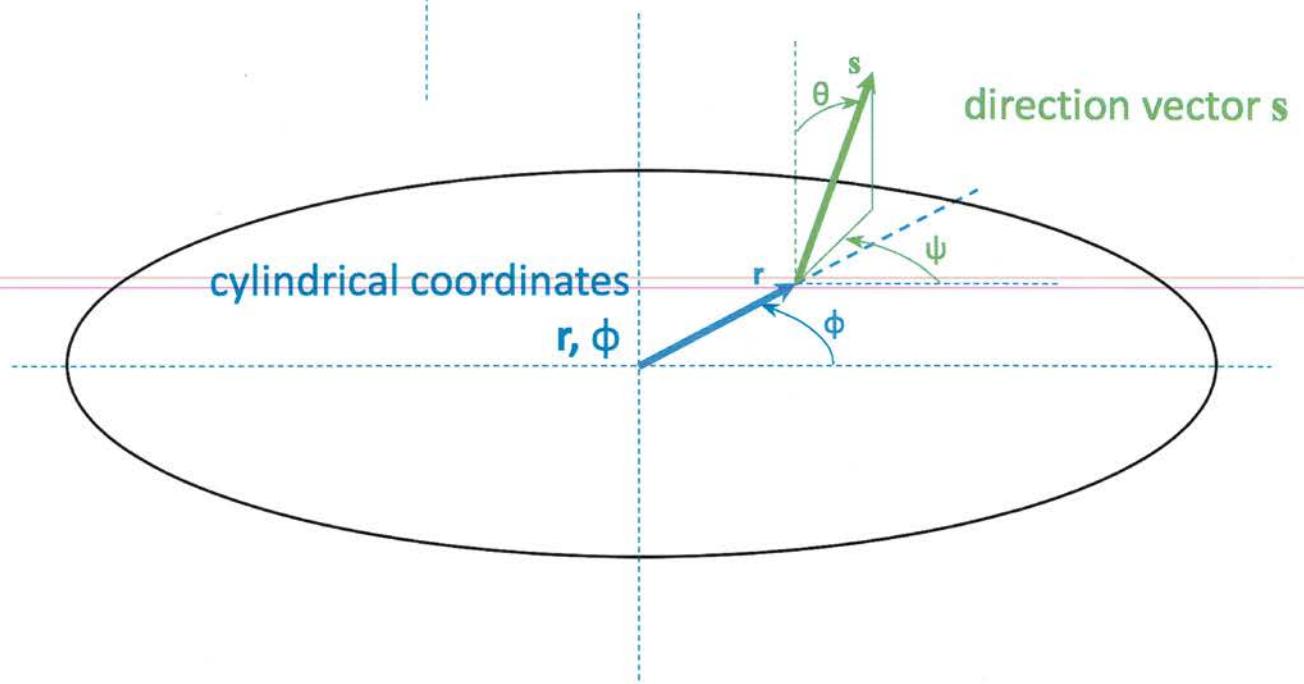
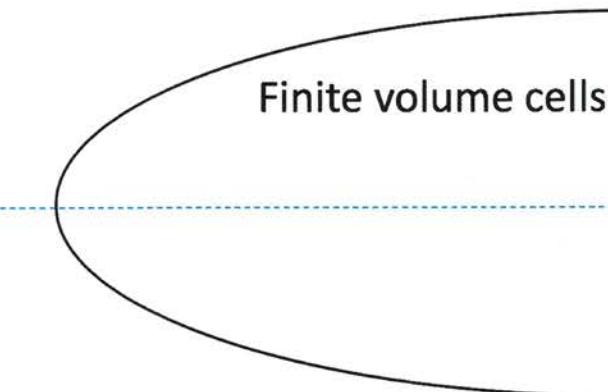


FVDOM

(1)

• Axisymmetric



$$\hat{S} \cdot \nabla I = K \bar{I}_b - K I = \text{RHS}.$$

\hat{S} is indep of ∇

$$\int_V \hat{S} \cdot \nabla I dV = \int_V \nabla \cdot \hat{S} I dV = \int_A \hat{S} I \cdot \hat{n} dA = \int_V \text{RHS} dV$$

$$\text{F.V.} \rightarrow \hat{S} \cdot n_e I_e A_e + \hat{S} \cdot n_w I_w A_w + \hat{S} \cdot n_n I_n A_n + \hat{S} \cdot n_s I_s A_s = \text{RHS} \cdot V$$

$$\cdot \hat{S} = \sin \theta \cos \psi \hat{i} + \sin \theta \sin \psi \hat{j} + (\cos \theta \hat{k})$$

$$\cdot \text{ Let } \alpha = \sin \theta \cos \psi \quad \beta = \sin \theta \sin \psi \quad \rightarrow \hat{S} = \alpha \hat{i} + \beta \hat{j}$$

(2)

$$\vec{n}_e = \cos(\phi) \vec{i} + \sin(\phi) \vec{j}$$

$$\vec{n}_w = \cos(\phi + \pi) \vec{i} + \sin(\phi + \pi) \vec{j}$$

$$\vec{n}_n = \cos\left(\phi + \frac{\Delta\phi}{2} + \frac{\pi}{2}\right) \vec{i} + \sin\left(\phi + \frac{\Delta\phi}{2} + \frac{\pi}{2}\right) \vec{j}$$

$$\vec{n}_s = \cos\left(\phi - \frac{\Delta\phi}{2} - \frac{\pi}{2}\right) \vec{i} + \sin\left(\phi - \frac{\Delta\phi}{2} - \frac{\pi}{2}\right) \vec{j}$$

use $\cos\left(\frac{\pi}{2} + \gamma\right) = -\sin\gamma$

$$\sin\left(\frac{\pi}{2} + \gamma\right) = \cos\gamma$$

$$\cos(\gamma) = \cos(-\gamma)$$

$$\sin(-\gamma) = -\sin(\gamma)$$

$$\vec{n}_e = \cos(\phi) \vec{i} + \sin(\phi) \vec{j}$$

$$\vec{n}_w = -\cos(\phi) \vec{i} + \sin(\phi) \vec{j}$$

$$\vec{n}_n = -\sin\left(\phi + \frac{\Delta\phi}{2}\right) \vec{i} + \cos\left(\phi + \frac{\Delta\phi}{2}\right) \vec{j}$$

$$\vec{n}_s = \sin\left(\phi - \frac{\Delta\phi}{2}\right) \vec{i} - \cos\left(\phi - \frac{\Delta\phi}{2}\right) \vec{j}$$

Take $\Delta\phi$ as the angular width of the control volumes

Consider volumes for which $\phi = 0$

$$\vec{n}_e = \vec{i}$$

$$\vec{n}_w = -\vec{i}$$

$$\vec{n}_n = -\sin\left(\frac{\Delta\phi}{2}\right) \vec{i} + \cos\left(\frac{\Delta\phi}{2}\right) \vec{j}$$

$$\vec{n}_s = -\sin\left(\frac{\Delta\phi}{2}\right) \vec{i} - \cos\left(\frac{\Delta\phi}{2}\right) \vec{j}$$

$$\hat{s} = \alpha \vec{i} + \beta \vec{j}$$

$$\alpha = \sin\theta \cos\psi$$

$$\beta = \sin\theta \sin\psi$$

$$FV \rightarrow \hat{s} \cdot n_e I_e A_e + \hat{s} \cdot n_w I_w A_w + \hat{s} \cdot n_n I_n A_n + \hat{s} \cdot n_s I_s A_s = RHS \cdot V$$

$$\alpha I_e A_e - \alpha I_w A_w + I_n A_n \left(-\alpha \sin\left(\frac{\Delta\phi}{2}\right) + \beta \cos\left(\frac{\Delta\phi}{2}\right) \right)$$

$$+ I_s A_s \left(-\alpha \sin\left(\frac{\Delta\phi}{2}\right) - \beta \cos\left(\frac{\Delta\phi}{2}\right) \right) = RHS \cdot V$$

$$A_e = r_e \Delta\phi$$

$$A_w = r_w \Delta\phi$$

$$A_s = r_e - r_w$$

$$A_n = r_e - r_w$$

$$V = \frac{1}{2} \Delta\phi (r_e^2 - r_w^2)$$

(3)

$$\alpha I_e r_e \Delta\phi - \alpha I_w r_w \Delta\phi + I_n (r_e - r_w) \left(-\alpha \sin \frac{\Delta\phi}{2} + \beta \cos \frac{\Delta\phi}{2} \right) + I_s (r_e - r_w) \left(-\alpha \sin \frac{\Delta\phi}{2} - \beta \cos \frac{\Delta\phi}{2} \right) = \frac{1}{2} \Delta\phi (r_e^2 - r_w^2) \text{ RHS}$$

 $\div \Delta\phi$

$$\frac{\alpha I_e r_e - \alpha I_w r_w - \alpha (r_e - r_w) \sin \frac{\Delta\phi}{2} (I_n + I_s)}{\Delta\phi} + \frac{\beta (r_e - r_w) \cos \frac{\Delta\phi}{2} (I_n - I_s)}{\Delta\phi} = \frac{1}{2} (r_e^2 - r_w^2) \text{ RHS}$$

Term 1 Term 2

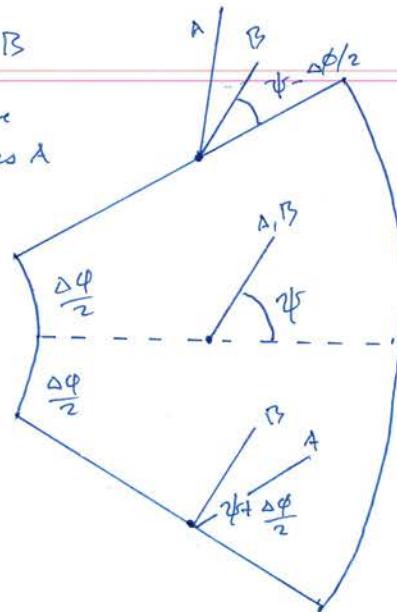
- Symmetry wrt ψ implies

$$I_{n,\psi} = I_n(\theta_\psi, \psi_\psi) = I_p(\theta_\psi, \psi_\psi - \frac{\Delta\phi}{2})$$

$$I_{s,\psi} = I_s(\theta_\psi, \psi_\psi) = I_p(\theta_\psi, \psi_\psi + \frac{\Delta\phi}{2})$$

- I_p is I at cell center P

- Direction ψ is lines B
- Symmetry gives same intensity along lines A



- All B lines are parallel
- All A lines make the same angle with the respective faces

Term 2

$$\frac{\beta (r_e - r_w) \cos \frac{\Delta\phi}{2} (I(\theta_\psi, \psi - \frac{\Delta\phi}{2}) - I(\theta_\psi, \psi + \frac{\Delta\phi}{2}))}{\Delta\phi}$$

- let $\Delta\phi \equiv \Delta\psi$
- Both are measured from \vec{i} $\Rightarrow \Delta\phi$ is just some angle

$$\lim_{\Delta\phi \rightarrow 0} ; \cos \frac{\Delta\phi}{2} = \cos 0 = 1$$

$$\rightarrow \boxed{-\beta (r_e - r_w) \frac{dI}{d\psi}}$$

(4)

Term 1

$$\frac{-\alpha(r_e - r_\omega) \sin \frac{\Delta\phi}{2} \left(I(\theta, \psi - \frac{\Delta\phi}{2}) + I(\theta, \psi + \frac{\Delta\phi}{2}) \right)}{\Delta\phi}$$

$$\lim_{\Delta\phi \rightarrow 0} \frac{0}{0}$$

Use L'Hopital's rule: take $\frac{d}{d\Delta\phi}$ of the numerator & Denominator.

• Denominator $\rightarrow 1$

• Numerator: $-\alpha(r_e - r_\omega) \left[\left((I(\theta, \psi - \frac{\Delta\phi}{2}) + I(\theta, \psi + \frac{\Delta\phi}{2})) \cdot \frac{1}{2} \cos \frac{\Delta\phi}{2} + \right) \sin \frac{\Delta\phi}{2} \left(\frac{dI(\theta, \psi - \frac{\Delta\phi}{2})}{d\Delta\phi} + \frac{dI(\theta, \psi + \frac{\Delta\phi}{2})}{d\Delta\phi} \right) \right]$

$\underbrace{\qquad\qquad\qquad}_{=0}$

$$\frac{dI(\gamma)}{d\gamma} = \frac{dI(-\gamma)}{d(-\gamma)} = \frac{dI(-\gamma)}{d\gamma} \cdot \frac{dy}{d(-\gamma)} = -\frac{dI(-\gamma)}{d\gamma}$$

$$\rightarrow -\alpha(r_e - r_\omega) \cdot \frac{1}{2} \cos \frac{\Delta\phi}{2} \cdot (I(\theta, \psi - \frac{\Delta\phi}{2}) + I(\theta, \psi + \frac{\Delta\phi}{2}))$$

$$\lim_{\Delta\phi \rightarrow 0} : -\alpha(r_e - r_\omega) \cdot I(\theta, \psi)$$

Term 1 \rightarrow - $\alpha(r_e - r_\omega) I$

* FV $\rightarrow \alpha I_e r_e - \alpha I_{\omega r} - \alpha(r_e - r_\omega) I_p - \beta(r_e - r_\omega) \frac{dI}{d\psi} = \frac{1}{2}(r_e^2 - r_\omega^2) \cdot \text{RHS.}$ *

• Compare to Modest 17.30, 17.31. which is Cylindrical DOM, Differential form.

• $\div(r_e - r_\omega)$, $\lim_{r_e - r_\omega \rightarrow 0} :$

$$\alpha \frac{I_e r_e - I_{\omega r}}{r_e - r_\omega} - \alpha I_p - \beta \frac{dI}{d\psi} = \frac{1}{2} \frac{(r_e^2 - r_\omega^2)}{r_e - r_\omega} \text{ RHS} = \frac{1}{2} \frac{(r_e/r_\omega)(r_e + r_\omega)}{(r_e/r_\omega)} \text{ RHS}$$

$$\alpha \frac{dI_r}{dr} - \alpha I - \beta \frac{dI}{d\psi} = r \cdot \text{RHS.}$$

$$\underbrace{-\frac{\partial I \beta}{\partial \psi}}$$

= Modest 17.31

$\omega / \alpha \rightarrow \mu$,
 $\beta \rightarrow M$,
 $\div r$

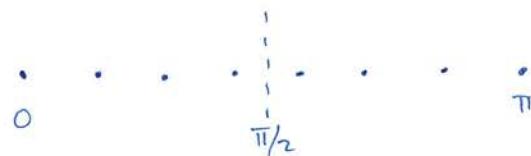
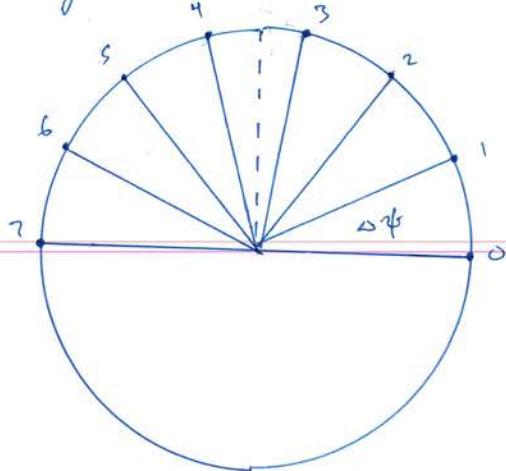
(S)

$$\alpha I_{ee} - \alpha I_{\omega r\omega} - \alpha(r_e - r_\omega) I_p - \beta(r_e - r_\omega) \frac{dI}{d\psi} = \frac{1}{2}(r_e^2 - r_\omega^2)(kI_b - kI)$$

- $\alpha = \sin \theta \cos \psi$
- $\beta = \sin \theta \sin \psi$
- All I, α, β are for a Particular Direction i

Discretize ψ, θ rewrite Direction i as an i, j pair: θ_i, ψ_j

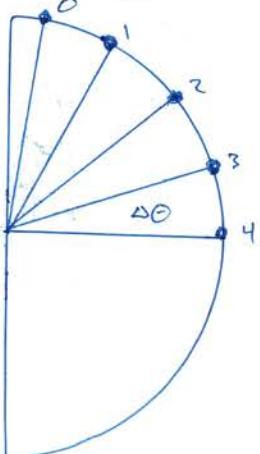
- $\psi_j : 0 \rightarrow \pi$



$$\Delta\psi = \frac{\pi}{n_\psi-1}$$

use even n_ψ For upwind "symmetry"

- $\theta_i : 0 \rightarrow \frac{\pi}{2}$



$$\Delta\theta = \frac{\frac{\pi}{2}}{n_\theta-1}$$

$$\Delta\theta = \frac{\frac{\pi}{2}}{n_\theta-1}$$

exclude 0 else $\alpha = \beta = 0$

Note $\left. \frac{dI}{d\psi} \right|_{\psi=\psi_0=0} = 0$

By Symmetry.

for $\psi=0$, \hat{s} is purely radial ~~Maxwell~~
and $I(\theta, 0 + \Delta\psi) = I(\theta, 0 - \Delta\psi)$
 $\Rightarrow dI/d\psi = 0$

(6)

$$\alpha_{xy} I_e^y r_e - \alpha_{xy} I_w^y r_w - \alpha_y (r_e - r_w) I_p^{y'} - \beta (r_e - r_w) \frac{dI}{d\psi} = \frac{1}{2} (r_e^2 - r_w^2) (K_I b - K_p I_p^{y'})$$

- Note $\frac{dI}{d\psi}$ couples the directions. / (for given i)
- But since $\left. \frac{dI}{d\psi} \right|_{x,0} = 0$, we can first solve for $j=0$, then for other $j > 0$ we can backward difference.

$$i, j=0: \frac{dI}{d\psi} = 0$$

$$i, j > 0: \frac{dI}{d\psi} = \frac{I^j - I^{j-1}}{\Delta\psi}$$

- upwind face I's:

for $0 \leq \psi \leq \frac{\pi}{2}$ I is

$$I_e = I_E$$

$$I_w = I_P$$

for $\frac{\pi}{2} \leq \psi \leq \pi$ I is

$$I_e = I_P$$

$$I_w = I_W$$